

CALCULUS AND DIFFERENTIAL EQUATIONS WITH AI

$$\int \sin^2 x \, dx$$

$$y'' + 7y' + 3y = \cos x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 1$$



Pramote Dechaumphai

Calculus and Differential Equations with AI

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Preface

Artificial Intelligence (AI) is playing an increasingly important role in education, especially in the areas of mathematics and computer-based problem solving. Today, AI is capable of solving calculus and differential equation problems using symbolic mathematical methods, following step-by-step procedures that closely resemble those taught in the classroom. In addition, AI can rapidly generate code to visualize results and graphs with clarity and colorful presentation.

However, it is important to note that AI often comes with a disclaimer: its answers may not always be accurate. This raises a concern that students may depend on AI-generated solutions without fully grasping the underlying concepts. To address this, *Calculus and Differential Equations with AI* was written to encourage thoughtful and responsible use of AI in learning. The book presents conventional mathematical methods first, then compares them with AI-generated solutions using concise commands.

This approach allows readers to critically evaluate the accuracy of AI outputs while developing a strong, step-by-step understanding of mathematical reasoning. At the same time, it demonstrates how AI can be a powerful tool to support learning and problem-solving efficiently and effectively.

The author would like to express sincere thanks to Chulalongkorn University Press for their generous support in the publication and distribution of this book. It is the author's hope that this book will help students build a deeper understanding of both mathematics and AI, and inspire them to explore the exciting possibilities of applying AI to more advanced challenges in the future.

Pramote Dechaumphai

***** Precautions for Using AI *****

Mathematics is the subject that requires deep understanding and consistent self-practice. This book is designed to help developing these skills through clear, step-by-step explanations. It includes examples of fundamental mathematical problems along with detailed solution procedures. Therefore, it is important to always keep in mind that:

- It's helpful if we try solving the problems on our own first, and then use AI to review the answers.
- When we encounter difficulties along the way, we can ask AI for clarification to better understand each step.
- By practicing how to ask meaningful questions, we can use AI to deepen our understanding of where the results come from.
- While AI can provide quick answers, it's important that we take time to think through the problems ourselves, because independent thinking is at the heart of learning mathematics.
- AI can sometimes make mistakes, so we should always verify the answers instead of accepting them right away.
- The exercises at the end of each chapter are a great opportunity for us to strengthen our understanding. They can make learning more enjoyable, and the math skills we develop will be valuable for solving real-world problems.

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Chapter 1

Symbolic Mathematics

1.1 Introduction

Artificial Intelligence (AI), such as ChatGPT and DeepSeek, is increasingly playing a role in mathematics education today. A solid understanding of mathematics is important because it is essential for analyzing problems in science and engineering. For this reason, students in many educational institutions are required to study fundamental mathematics courses such as *Calculus*, *Differential Equations*, and other related subjects. These courses provide the necessary mathematical foundation for solving more advanced problems.

Mathematics education typically begins with learning about different types of *functions*, followed by concepts like *limit*, *differentiation*, and *integration*. Each of these topics is complex and requires specific understanding. As a result, students often memorize procedures to prepare for exams, but many tend to forget them afterward.

Today, AI tools can quickly provide answers to basic mathematical problems. However, the results from AI must still be verified for accuracy, and students should understand the steps and reasoning behind the answers.

AI tools used to solve symbolic mathematics problems, the kind we often study in class, are usually built on top of other software. For example, OpenAI's ChatGPT often relies on software like Wolfram Alpha, also known as Mathematica, which has been trusted for many years. Other AI systems may use their own built-in software or call on different external tools. Because of this, it is important for users to verify the correctness of the results themselves.

For instance, when we use AI to solve an *ordinary differential equation (ODE)*, how can we be sure that the answer is correct? By checking the result through other methods or understanding the steps involved, users can better grasp how the solution was obtained. This not only ensures accuracy but also deepens their understanding of mathematics.

These AI tools can display the results directly in symbolic form on the computer screen. For example:

$$\int x^2 dx = \frac{x^3}{3}$$

It is also important to note that solving problems in science and engineering often requires results in numerical form. For example, when we want to compute an integral,

$$\int_0^3 x^2 dx$$

AI can instantly provide a numerical result, such as the value 9. Moreover, if the *integrand* (the function to be integrated) is complex, AI can generate a computer program using common programming languages like Python or MATLAB. This allows users to conveniently perform the final computation on a portable computer such as a notebook.

1.2 Solving Mathematical Equations with AI

In learning calculus, we need to memorize various formulas to find the *derivatives* and *integrals* of functions. If we remember the formulas correctly and follow each step carefully, we can obtain accurate results. The following examples illustrate this process.

Example Find the first derivative of the function,

$$f = \frac{x}{3 - 4\cos x}$$

We can easily obtain the result by using ChatGPT or DeepSeek, simply by typing:

Determine the first derivative of $f = x/(3-4*\cos(x))$.

Both ChatGPT and DeepSeek will show the step-by-step process for finding the first derivative of the function f directly on the computer screen, as shown below.

To find the first derivative of the function:

$$f(x) = \frac{x}{3 - 4\cos(x)}$$

we will use the Quotient Rule for differentiation. The Quotient Rule states that if we have a function $\frac{u(x)}{v(x)}$, its derivative is:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

Here, let: $u = x$ and $v = 3 - 4\cos(x)$.

Step 1 Compute u' and v' :

Derivative of $u = x$ is: $u' = \frac{d}{dx}(x) = 1$

Derivative of $v = 3 - 4\cos(x)$ is: $v' = \frac{d}{dx}(3 - 4\cos(x)) = 0 - 4(-\sin(x)) = 4\sin(x)$

Step 2 Apply the Quotient Rule:

Substituting u , u' , v and v' into the Quotient Rule formula:

$$f'(x) = \frac{u'v - uv'}{v^2} = \frac{(1)(3 - 4\cos(x)) - (x)(4\sin(x))}{(3 - 4\cos(x))^2}$$

Simplify the numerator to obtain the final answer:

$$f'(x) = \frac{3 - 4\cos(x) - 4x\sin(x)}{(3 - 4\cos(x))^2}$$

Example Find the derivatives of a more complex function,

$$g = \frac{x \tan x - 3 \tan x - 21x^3 + 7x^4}{x^3 - 3x^2 + 6x - 18}$$

Similarly, the first derivative can be easily obtained using AI, resulting in:

$$\frac{dg}{dx} = -\frac{2x \tan x + 504}{(x^2 + 6)^2} + \frac{\tan^2 x + 43}{x^2 + 6} + 7$$

and the second derivative of this function is:

$$\frac{d^2g}{dx^2} = \frac{2016x - 48 \tan x}{(x^2 + 6)^3} - \frac{4x \tan^2 x - 6 \tan x + 88x}{(x^2 + 6)^2} + \frac{2 \tan x (\tan^2 x + 1)}{x^2 + 6}$$

Such complex results can easily lead to mistakes if done by hand, as we are used to doing in the classroom.

Example Integration is another major topic in calculus that often causes discomfort for learners. This is because it requires memorizing many formulas and techniques.

Additionally, students may not clearly see the practical value or how the results can be applied in real-world problems. For example,

$$\int \frac{dx}{x^2 + 9} = \frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right) + C$$

Here, C is the constant of integration. In cases like this, we can use ChatGPT or DeepSeek by simply typing:

Integrate $dx/(x^2+9)$.

Both ChatGPT and DeepSeek will display the step-by-step solution on the computer screen, as shown below.

To evaluate the integral $\int \frac{dx}{x^2 + 9}$, we use the standard formula:

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \left(\frac{x}{a} \right) + C$$

In our case, $a^2 = 9$ so $a = 3$. Thus,

$$\int \frac{dx}{x^2 + 9} = \frac{1}{3} \arctan \left(\frac{x}{3} \right) + C$$

Here is another example that is more complex. Both ChatGPT and DeepSeek can still provide the correct result, as shown below:

$$\int_1^2 \frac{x dx}{(x+1)(2x-1)} = \frac{1}{2} \ln(3) - \frac{1}{3} \ln(2)$$

Example Symbolic mathematics in AI can also help solving other problems that would normally take a long time for most people. For example, if we want to factor a function,

$$h = x^4 + 26x^3 - 212x^2 - 1578x + 5859$$

We can use AI to perform the task, which produces the following result,

$$h = (x+7)(x-9)(x-3)(x+31)$$

Herein, we can use DeepSeek by typing the following command:

Factor $x^4 + 26*x^3 - 212*x^2 - 1578*x + 5859$.

DeepSeek will display the factoring process in a detailed, step-by-step manner. It also verifies the correctness of the result. In the end, it summarizes the final result as follows:

After systematically applying the Rational Root Theorem and synthetic division, I was able to factor the polynomial completely. The final factored form is,

$$h = (x-3)(x-9)(x+7)(x+31)$$

Example Differential Equation is another subject that many students find challenging, often comparing it to taking bitter medicine. This is because it requires memorizing specific techniques and procedures to solve different types of differential equations. For example:

$$\frac{dy}{dx} + y = 5$$

The solution to the above first-order differential equation is:

$$y(x) = Ce^{-x} + 5$$

where C is a constant that can be determined using the initial condition provided in the problem.

The above solution can be easily obtained using DeepSeek by typing the following:

Solve the differential equation: $dy/dx + y = 5$.

DeepSeek will show the step-by-step solution and provide the final answer as follows.

The general solution to the differential equation $\frac{dy}{dx} + y = 5$ is:

$$y(x) = Ce^{-x} + 5$$

If the given equation is more complex, such as in the form of a second-order differential equation,

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$$

DeepSeek can provide the solution to this second-order differential equation as,

$$y(x) = C_1 e^{-x} + C_2 e^{-3x}$$

where C_1 and C_2 are constants that can also be determined from the specific initial conditions given in the problem.

The solutions of differential equations can be plotted directly on the computer screen, helping learners better understand the meaning of the phenomena

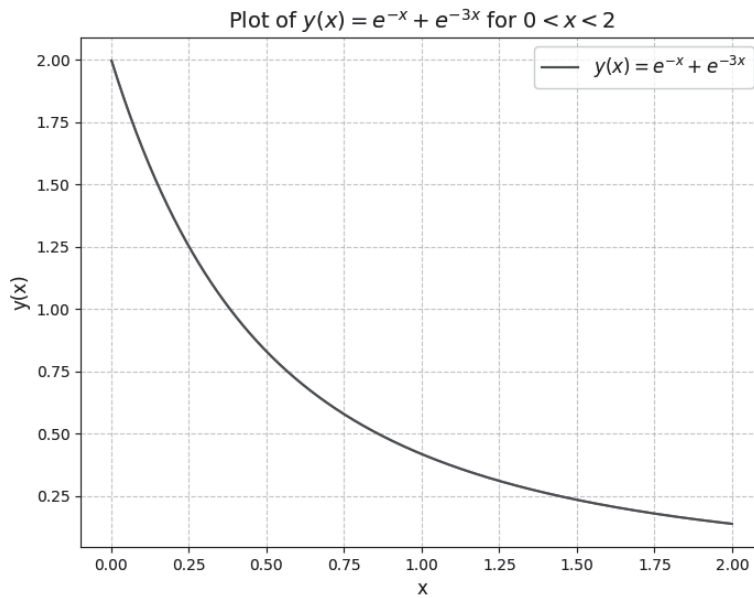
represented by these solutions. For example, if $C_1 = C_2 = 1$, then the solution to this second-order differential equation is,

$$y(x) = e^{-x} + e^{-3x}$$

AI tools like ChatGPT can plot this graph directly by simply typing the following command:

Plot $y(x) = \exp(-x) + \exp(-3x)$ for $0 < x < 2$.

AI will generate a plot of y as a function of x , as shown in the figure. Additionally, we can ask the AI to generate code in any programming language, such as Python. In this case, the code provided is:



```
import numpy as np
import matplotlib.pyplot as plt

# Define the function
def y(x):
    return np.exp(-x) + np.exp(-3 * x)

# Generate x values from 0 to 2 (exclusive) with small step size
x = np.linspace(0.001, 2, 400) # Avoid x=0 to prevent division issues

# Compute y values
y_values = y(x)

# Create the plot
plt.figure(figsize=(8, 6))
plt.plot(x, y_values, label=r'$y(x) = e^{-x} + e^{-3x}$', color='blue')

# Add labels and title
plt.xlabel('x', fontsize=12)
plt.ylabel('y(x)', fontsize=12)
plt.title('Plot of $y(x) = e^{-x} + e^{-3x}$ for $0 < x < 2$', fontsize=14)
```

```
# Add grid and legend
plt.grid(True, linestyle='--', alpha=0.6)
plt.legend(fontsize=12)

# Show the plot
plt.show()
```

1.3 Precautions When Using AI

Using AI to solve problems in Calculus and Differential Equations can be very helpful, but it is important to be aware of certain errors and limitations, as outlined below:

(a) Accuracy of the answer

- AI may provide incorrect answers, especially for complex problems or when the input question contains errors.
- Users should verify the answers themselves or use multiple tools to cross-check the results, such as Wolfram Alpha and Maxima.
- Users should be cautious when calculating limits, integrals, or derivatives that involve special conditions—for example, *improper integrals*, where the limits of integration extend to infinity, or cases where the integrand diverges and does not produce a finite result.

(b) Step-by-step solutions

- Some AI tools only show the final answer without displaying the step-by-step process, which makes it difficult for learners to understand and learn from the solution.
- Choose AI tools that provide detailed step-by-step solutions, so you can verify the accuracy and better understand the underlying concepts.

(c) Problem interpretation

- AI may misinterpret the problem if the question is ambiguous or uses non-standard symbols.
- Write the problem clearly, and make sure the question matches what you actually want to ask.

(d) Mathematical limitations

- AI may not be able to solve problems that require certain types of reasoning or intuition that come naturally to humans.
- Problems related to existence and uniqueness in differential equations may be overlooked.

- Calculations involving series solutions, Laplace transforms, or solving partial differential equations (PDEs) may lead to errors if the problems are highly complex.
- (e) Checking assumptions
- When solving differential equations, AI may not thoroughly check initial conditions or boundary conditions, which can lead to incomplete or incorrect solutions.
 - Users should verify that the obtained solution satisfies the given initial or boundary conditions.
- (f) Over-reliance
- Relying on AI to solve problems all the time may prevent users from truly understanding the underlying concepts.
 - Therefore, AI should be used as a learning aid—not as a replacement for practicing and solving problems on your own.
- (g) Technical errors
- Sometimes, AI may provide answers that seem reasonable but are mathematically incorrect, a phenomenon known as hallucination.
 - Therefore, users should verify the results by hand or with other reliable tools.
- (h) Real-world applications
- Calculus and differential equation problems in real-world applications are often more complex than those found in textbooks. AI may not be able to fully interpret or apply the real-world context of such problems.
- (i) References and sources
- Some AI tools do not show the steps or cite the formulas and theories used, making it difficult to verify the correctness of the results.
- (j) Therefore, AI users should:
- Use AI tools such as Wolfram Alpha, ChatGPT with plugins, Symbolab, or Maxima to verify the correctness of the results.
 - Practice solving problems on your own first, then use AI to check your answers.
 - Ask teachers or classmates if the answer provided by AI seems questionable. Combining this process with developing your own understanding will lead to the greatest learning benefit.

1.4 Conclusion

In this chapter, we have learned about the importance and benefits of using AI tools such as ChatGPT and DeepSeek to solve simple mathematical problems. These tools can significantly reduce the difficulty and time required to solve problems by hand. They also help minimize errors, especially when dealing with complex mathematical expressions. Most importantly, AI allows us to spend more time studying the results, understanding their physical meaning through plotting, and visually interpreting the behavior of solutions directly on the computer screen.

AI is an extremely useful tool, but it can sometimes produce incorrect results. Therefore, AI users must verify the accuracy of the answers obtained. This verification process not only ensures the correctness of the results but also deepens the learner's understanding of mathematical problem-solving. In the following chapters, we will explore methods for checking the results generated by AI to ensure their accuracy and reliability.

Exercises

The exercises in this chapter focus on understanding the characteristics of functions, including expansion, simplification, and plotting functions in various forms using appropriate commands. AI tools like ChatGPT and DeepSeek will present detailed step-by-step solutions that lead to the final result. The methods used to obtain these results may differ from those traditionally taught in class, so students are encouraged to explore and study them to gain a broader and more diverse understanding.

1. Create commands for AI tools such as ChatGPT or DeepSeek to find the following values:

(a) $(-3)^4$

(b) $16^{-3/4}$

(c) $3(x+6) + 4(2x-7)$

(d) $(x-3)(4x+1)$

(e) $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$

(f) $3x^{3/2} - 9x^{1/2} + 6x^{-1/2}$

2. Create commands for AI tools such as ChatGPT or DeepSeek to simplify or evaluate the following expressions:

$$(a) \frac{x^2 + 3x + 2}{x^2 - x - 2}$$

$$(b) \frac{2x^2 - x - 1}{x^2 - 9} \cdot \frac{x + 3}{2x + 1}$$

$$(c) \frac{x^2}{x^2 - 4} - \frac{x + 1}{x - 2}$$

$$(d) \frac{\frac{y}{x} - \frac{x}{y}}{\frac{1}{y} - \frac{1}{x}}$$

$$(e) \left(\frac{3x^{3/2}y^3}{x^2y^{-1/2}} \right)^{-2}$$

$$(f) \frac{(4x^3y^3)(3xy^2)^2}{\sqrt{xy}}$$

3. Use AI tools such as ChatGPT or DeepSeek to find the product of the functions f and g in each of the following problems:

$$(a) f = x^2 - 2x \quad ; \quad g = (x + 1)^2$$

$$(b) f = \sqrt{x - 1} \quad ; \quad g = \sqrt{x^2 + 2}$$

$$(c) f = \sin(x - x^2) \quad ; \quad g = \cos(x^2 + x)$$

$$(d) f = e^{x-2} \quad ; \quad g = \ln(x + 3)$$

$$(e) f = \sin(x^2 - x) \quad ; \quad g = \ln(x^2 + 2x)$$

$$(f) f = e^x \quad ; \quad g = e^{-2x^2}$$

4. Create commands in AI tools such as ChatGPT or DeepSeek to simplify the following functions into their simplest form.

$$(a) f = 6x^4 + 28x^3 - 7x^2 + 14x - 5$$

$$(b) g = (\cos x - \sin x)(\cos x + \sin x)(e^x + \sin x)(3x - 7)$$

$$(c) h = \frac{x \tan x - 3 \tan x - 21x^3 + 7x^4}{x^3 - 3x^2 + 6x - 18}$$

$$(d) u = 3x + x^2 + 4\sqrt{x} + 2x\sqrt{x} + 2$$

$$(e) v = \frac{\sin x \tan^2 x + 6 \sin x \tan x + 9 \sin x}{x \cos^2 x - x - 7 \cos^2 x + 7}$$

$$(f) w = \frac{15x + 3xy^2 + 5y^2 + y^4}{x^2(\cos x \sin y - \cos y \sin x) - \cos x \sin y - \cos y \sin x}$$

5. Create commands in AI tools such as ChatGPT or DeepSeek to prove that:

$$(a) \quad x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$$

$$(b) \quad \sin(3x) = 3\sin x - 4\sin^3 x$$

$$(c) \quad \tan(4x) = \frac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x}$$

$$(d) \quad \cos^4 x = \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)$$

$$(e) \quad (e^x + e^{-x})\tanh x = e^x - e^{-x}$$

6. Create commands in AI tools such as ChatGPT or DeepSeek to prove that:

$$(a) \quad \tanh^2 x + \operatorname{sech}^2 x = 1$$

$$(b) \quad \tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$$

$$(c) \quad \cosh \frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$$

$$(d) \quad \cosh(4x) = 8\cosh^4 x - 8\cosh^2 x + 1$$

$$(e) \quad \sinh^3 x = \frac{1}{4}\sinh(3x) - \frac{3}{4}\sinh x$$

7. Use AI tools such as ChatGPT or DeepSeek to plot the following functions. Specify an appropriate range and scale for the plot.

$$(a) \quad f = 3x^5 - 25x^3 + 60x$$

$$(b) \quad g = \frac{5x^2 + 8x - 3}{3x^2 + 2}$$

$$(c) \quad h = \frac{3}{4}(x^2 - 1)^{2/3}$$

$$(d) \quad u = \sin(\pi/x)$$

$$(e) \quad v = (x - 2)^4(x + 1)^3(x - 1)$$

8. Use AI tools such as ChatGPT or DeepSeek to plot the following functions. Specify an appropriate range and scale for the plot.

$$(a) \quad f = x^3 - x + 1$$

$$(b) \quad g = x^4 - 3x^2 + x$$

$$(c) \quad h = 3x^5 - 25x^3 + 60x$$

9. Type the following command into AI tools such as ChatGPT or DeepSeek: *"Plot any function that looks visually striking."* Then, examine the result. The output of this unusual function clearly demonstrates the usefulness of functions in mathematical applications.
10. Type the following command into AI tools such as ChatGPT or DeepSeek: *"Plot any function that exhibits a sudden change."* Then, examine the result. Functions that exhibit sudden changes often arise from solving hyperbolic differential equations, which are commonly found in real-world problems, such as shock waves in front of a supersonic aircraft or shock waves generated by a nuclear explosion.
11. Type the following command into AI tools such as ChatGPT or DeepSeek: *"Plot a beautiful 3D surface function."* Then, observe the result. Functions that change gradually, like those shown in the generated surface, often arise from solving elliptic differential equations. These types of equations are common in real-world applications. For examples, temperature distribution inside a microchip, or stress distribution in a roof structure.

Chapter 2

Calculus

2.1 Introduction

Calculus is a fundamental subject that all science and technology students are required to study. The core of calculus consists of two main parts: differentiation and integration, both of which rely on a good understanding of functions and limits. Learning calculus, especially integrating complex functions, can be challenging for many students. It often feels difficult because it requires memorizing various formulas and steps in order to arrive at the correct answers.

The capability of symbolic mathematics in AI allows us to obtain results quickly. In this chapter, we will explore the step-by-step process of solving problems that involve important concepts in calculus before using AI to find the same results. The purpose is to help students understand and appreciate the problem-solving process and recognize how modern symbolic AI can now assist in solving calculus problems with ease.

2.2 Limit

Finding the limit of a function $f(x)$ as a variable x approaches a certain value a can be written in the form of an equation as,

$$L = \lim_{x \rightarrow a} f(x)$$

Example Given,

$$f(x) = x^2$$

Then,

$$L = \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} x^2 = a^2$$

We can find the limit of this function by using a natural language command with an AI tool like DeepSeek. For example:

Find limit of $L = x^2$ as x approaches a .

DeepSeek will interpret this request and provide the solution quickly by displaying a response on the computer screen like this:

To find the limit of x^2 as x approaches a , we can evaluate the function at $x = a$ directly, since x^2 is a continuous function for all real numbers.

$$\lim_{x \rightarrow a} x^2 = a^2$$

Example Given,

$$g(x) = \frac{x^2 - a^2}{x - a}$$

If we use a simple method like in the previous example,

$$L = \lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} \frac{a^2 - a^2}{a - a} = \frac{0}{0}$$

However, this method may not always work and can sometimes give an incorrect result. For example, if $a = 1$, then,

$$g(x) = \frac{x^2 - 1^2}{x - 1}$$

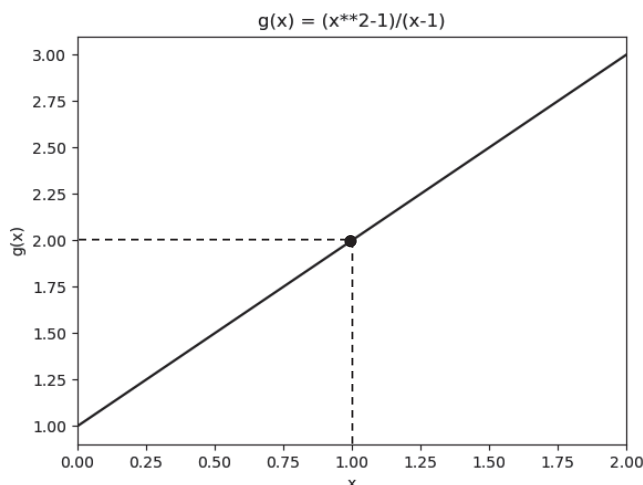
When we plot the function $g(x)$, as shown in the figure, we observe that as x approaches 1, the value of the function g approaches 2. This confirms that the limit of the function as x approaches 1 is indeed 2.

We can find the limit of this function by using a command on DeepSeek by typing:

Find limit of $g(x) = (x^2 - a^2)/(x - a)$ as x approaches a .

DeepSeek will then calculate and display the result automatically as follows.

To find the limit of the function $g(x) = \frac{x^2 - a^2}{x - a}$ as x approaches a , we can proceed with the following steps:



Step 1: Direct Substitution (Check for Indeterminate Form)

First, substitute $x = a$ directly into the function:

$$g(a) = \frac{a^2 - a^2}{a - a} = \frac{0}{0}$$

This results in the indeterminate form $0/0$, which means we cannot determine the limit directly and need to simplify the expression further.

Step 2: Simplify the Expression

Notice that the numerator $x^2 - a^2$ is a difference of squares, which can be factored as $x^2 - a^2 = (x - a)(x + a)$. Substituting this back into the original function,

$$g(x) = \frac{(x - a)(x + a)}{x - a}$$

For $x \neq a$, the $(x - a)$ terms cancel out, then,

$$g(x) = x + a$$

Step 3: Evaluate the simplified expression as x to a

Now, the simplified form $g(x) = x + a$ is continuous everywhere, including at $x = a$. Therefore, we can substitute $x = a$ directly,

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} (x + a) = a + a = 2a$$

The answer provided by DeepSeek is the same as the one we obtained earlier. We also find that DeepSeek follows the same reasoning process as we used at the beginning of this example. This gives us confidence in the result it produces.

Example Find the limit of,

$$L = \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$$

If we substitute $x = 4$ directly, we will get an undefined result of $0/0$ just like in the previous example. The idea for solving this type of problem is to multiply both the numerator and denominator by the conjugate of the numerator $\sqrt{x} + 2$ before evaluating the limit. The steps are as follows:

$$\begin{aligned} L &= \lim_{x \rightarrow 4} \frac{(\sqrt{x} - 2)}{(x - 4)} \cdot \frac{(\sqrt{x} + 2)}{(\sqrt{x} + 2)} = \lim_{x \rightarrow 4} \frac{(x - 4)}{(x - 4)(\sqrt{x} + 2)} \\ &= \lim_{x \rightarrow 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4} \end{aligned}$$

In the same way, if we use DeepSeek, we simply type in the expression, and we will get the same result as before.

Find limit of $L = (\text{sqrt}(x)-2)/(x-4)$ as x approaches 4.

To find the limit:

$$L = \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$$

we can proceed by rationalizing the numerator,

$$L = \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \lim_{x \rightarrow 4} \frac{(\sqrt{x})^2 - (2)^2}{(x - 4)(\sqrt{x} + 2)} = \lim_{x \rightarrow 4} \frac{x - 4}{(x - 4)(\sqrt{x} + 2)}$$

Simplify the numerator and cancel the common factor of $(x-4)$ to obtain,

$$L = \lim_{x \rightarrow 4} \frac{1}{\sqrt{x} + 2}$$

Now, substitute $x = 4$ into the simplified expression:

$$L = \frac{1}{\sqrt{4} + 2} = \frac{1}{2 + 2} = \frac{1}{4}$$

Example Find the limit of,

$$L = \lim_{x \rightarrow 0} \frac{\frac{1}{x+4} - \frac{1}{4}}{x}$$

In this case as well, if we try to directly substitute $x = 0$ into the function, we will get an undefined result of $0/0$. The techniques used in earlier examples,

such as factoring or multiplying both the numerator and denominator by the conjugate, cannot be applied here. To obtain the correct result, we need to multiply both the numerator and denominator by a suitable function, as shown below:

$$\begin{aligned} L &= \lim_{x \rightarrow 0} \frac{\left(\frac{1}{x+4} - \frac{1}{4}\right)}{x} \cdot \frac{4(x+4)}{4(x+4)} = \lim_{x \rightarrow 0} \frac{4 - (x+4)}{4x(x+4)} = \lim_{x \rightarrow 0} \frac{-x}{4x(x+4)} \\ &= \lim_{x \rightarrow 0} \frac{-1}{4(x+4)} = \frac{-1}{4(0+4)} = -\frac{1}{16} \end{aligned}$$

If we use DeepSeek, we can simply type the question as:

Find limit of $L = (1/(x+4)) - 1/4)/x$ as x approaches 0.

DeepSeek will show the steps as follows:

To find the limit of the expression:

$$L = \frac{\frac{1}{x+4} - \frac{1}{4}}{x}$$

as x approaches 0, we can follow these steps:

Step 1: Combine the Fractions in the Numerator.

First, combine the two fractions in the numerator over a common denominator,

$$\frac{1}{x+4} - \frac{1}{4} = \frac{4 - (x+4)}{4(x+4)} = \frac{-x}{4(x+4)}$$

Step 2: Simplify the Expression.

Now, substitute the simplified numerator back into the original expression,

$$L = \frac{\frac{-x}{4(x+4)}}{x} = \frac{-x}{4(x+4)} \cdot \frac{1}{x} = \frac{-1}{4(x+4)}$$

Step 3: Evaluate the limit as x approaches 0.

Finally, take the limit as x approaches 0,

$$\lim_{x \rightarrow 0} \frac{-1}{4(x+4)} = \frac{-1}{4(0+4)} = -\frac{1}{16}$$

which gives the same result.

We can see that AI provides the result of a function's limit quickly and conveniently, especially when the function becomes more complex, as shown in the following examples.

Example Find the limit of,

$$L = \lim_{x \rightarrow -4} \frac{4-x}{5-\sqrt{x^2-9}}$$

If we use DeepSeek, we simply type:

Find limit of L = (4-x)/(5-sqrt(x^2-9)) as x approaches -4.

The result obtained is:

$$\lim_{x \rightarrow -4} \frac{4-x}{5-\sqrt{x^2-9}} = \frac{8}{5-\sqrt{7}} = 3.398112$$

Example Find the limit of,

$$L = \lim_{x \rightarrow 0} \frac{e^{x^3} - 1}{1 - \cos \sqrt{x} - \sin x}$$

If we use DeepSeek, we simply type:

Find limit of L = (exp(x^3)-1)/(1-cos(sqrt(x-sin(x)))) as x approaches 0.

DeepSeek will display a rather lengthy step-by-step solution on the screen, which we will omit here. The final result is:

$$\lim_{x \rightarrow 0} \frac{e^{x^3} - 1}{1 - \cos \sqrt{x} - \sin x} = 12$$

Example Find the limit of the following function as x approaches infinity,

$$L = \lim_{x \rightarrow \infty} \frac{9x+4}{\sqrt{3x^2-5}}$$

If we use DeepSeek, we simply type:

Find limit of L = (9*x+4)/(sqrt(3*x^2-5)) as x approaches Infinity.

DeepSeek also displays a rather lengthy step-by-step solution on the screen, which we will omit here as well. The final result is:

$$\lim_{x \rightarrow \infty} \frac{9x+4}{\sqrt{3x^2-5}} = 3\sqrt{3} = 5.196152$$

Example If the function whose limit we want to find contains two variables x and y , DeepSeek can still find the result easily, e.g.,

$$L = \lim_{\substack{x \rightarrow -1 \\ y \rightarrow 2}} \frac{x^2y + xy^3}{(x+y)^3}$$

In this case, we can type:

Find limit of $L = (y \cdot x^2 + x \cdot y^3) / (x + y)^3$ as x approaches -1 and y approaches 2.

DeepSeek will display a rather lengthy step-by-step solution on the screen, which we will omit here. The result is:

$$\lim_{\substack{x \rightarrow -1 \\ y \rightarrow 2}} \frac{x^2 y + x y^3}{(x + y)^3} = -6$$

To gain a thorough understanding of the limit-solving process, learners should study the step-by-step solutions displayed on the screen. Understanding how to find the limit of a given function is a fundamental skill that forms the basis for learning how to differentiate functions in the following section.

2.3 Differentiation

Differentiation is a core topic in the study of calculus. It is essential because most scientific and engineering analyses begin with differential equations. A solid understanding of differentiation is therefore necessary and is required in many academic programs. However, finding the derivative of general or complex functions by hand, as commonly practiced in class, can be time-consuming and prone to error. The capability of symbolic mathematics in AI greatly reduces this complexity and helps eliminate many of these mistakes.

Let us begin with the definition of the derivative of y with respect to x , which is,

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Example Find the derivative of $y = f(x) = x^2$.

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} 2x + \Delta x \end{aligned}$$

$$\text{As } \Delta x \rightarrow 0, \quad \frac{dy}{dx} = 2x + 0 = 2x$$

We can use DeepSeek to find the derivative of this function $y = f(x) = x^2$ by typing a natural language command such as:

Find the derivative of $y = f(x) = x^2$ wrt x .

DeepSeek will display the steps for finding the result on the screen as follows:

To find the derivative of the function $y = f(x) = x^2$ with respect to x , we will use the power rule of differentiation. The power rule states that if $y = x^n$, then the derivative of y with respect to x is:

$$\frac{dy}{dx} = n \cdot x^{n-1}$$

Step-by-Step Solution:

Step 1: Identify the exponent n .

In the given function $y = x^2$, the exponent n is 2.

Step 2: Apply the power rule.

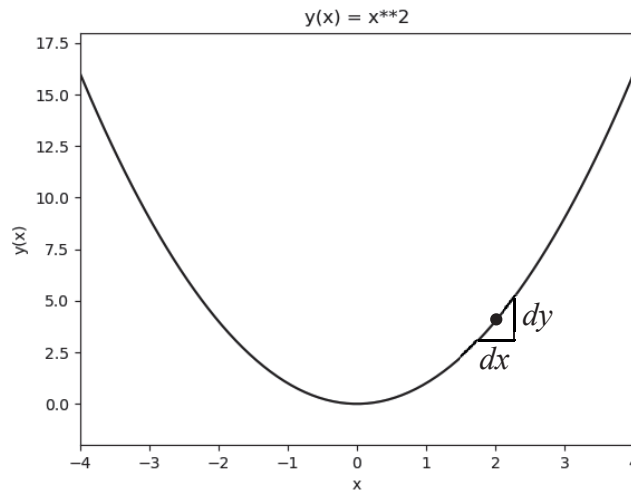
Multiply the exponent n by the variable x raised to the power of $n-1$:

$$\frac{dy}{dx} = 2 \cdot x^{2-1}$$

Step 3: Simplify the expression.

$$\frac{dy}{dx} = 2 \cdot x^1 = 2x$$

The behavior of the function $y = x^2$ is illustrated in the figure using the following set of Python commands:



```
import numpy as np
import matplotlib.pyplot as plt
x = np.linspace(-4, 4, 161)
y = x**2
plt.plot(x, y, 'k')
plt.xlabel('x'); plt.ylabel('y(x)')
plt.xlim(-4, 4); plt.ylim(-2, 18)
plt.title('y(x) = x**2')
plt.show()
```


The derivative dy/dx represents the slope at any point x . As an example at $x = 2$, $dy/dx = 2(2) = 4$.

Example Find the derivative of $y = f(x) = 2x - 3x^2$.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2(x + \Delta x) - 3(x + \Delta x)^2 - (2x - 3x^2)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2\Delta x - 6x\Delta x - 3(\Delta x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} 2 - 6x - 3\Delta x \\ &= \lim_{\Delta x \rightarrow 0} 2 - 6x - 3\Delta x = 2 - 6x - 3(0) = 2 - 6x\end{aligned}$$

Example Find the derivative when the given function is a constant, such as, $y = f(x) = 5$.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{5 - 5}{\Delta x} = \lim_{\Delta x \rightarrow 0} 0 \\ &= 0\end{aligned}$$

That is, the derivative of a constant function is equal to zero.

We can also use DeepSeek to find the derivative of the constant above by typing a natural language command such as:

Find the derivative of $y = f(x) = 5$ wrt x .

DeepSeek will display the steps for finding the result on the screen and give the answer of,

$$\frac{dy}{dx} = \frac{d}{dx}(5) = 0$$

Example Find the derivative of $y = f(x) = \sqrt{x}$ when $x > 0$.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} = \lim_{\Delta x \rightarrow 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}\end{aligned}$$

If we enter the command into DeepSeek as,

Find the derivative of $y = f(x) = \sqrt{x}$ wrt x for $x > 0$.

DeepSeek will display the step-by-step solution on the screen and provide the same answer, which is,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

Example Find the derivative of $y = f(x) = \frac{x}{2x+3}$.

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{x + \Delta x}{2(x + \Delta x) + 3} - \frac{x}{2x + 3}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\frac{x + \Delta x}{2(x + \Delta x) + 3} \cdot \frac{2x + 3}{2x + 3} - \frac{x}{2x + 3} \cdot \frac{2(x + \Delta x) + 3}{2(x + \Delta x) + 3}}{\Delta x} \end{aligned}$$

After performing manipulation,

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{3}{(2x + 2\Delta x + 3)(2x + 3)} = \frac{3}{(2x + 0 + 3)(2x + 3)}$$

i.e.,
$$\frac{dy}{dx} = \frac{3}{(2x + 3)^2}$$

We can easily obtain the same result by typing into DeepSeek as,

Find the derivative of $y = f(x) = x/(2*x+3)$ wrt x .

DeepSeek will display the steps for finding the result on the screen, possibly using the Quotient Rule as follows:

The Quotient Rule states that if you have a function $y = \frac{u}{v}$, where both u and v are functions of x , then the derivative of y with respect to x is,

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

which leads to the same result, that is,

$$\frac{dy}{dx} = \frac{3}{(2x + 3)^2}$$

In practice, the function $y = f(x)$ can be highly complex, making manual differentiation time-consuming and error-prone. We can conveniently use a command in DeepSeek to perform symbolic differentiation, as shown in the following examples.

Example Find the symbolic derivative of,

$$y = f(x) = x^4 - 8x^3 + 12x - 5$$

The, determines its derivatives at $x = -2, 0$, and 2 .

In this case, we may type into DeepSeek as follows,

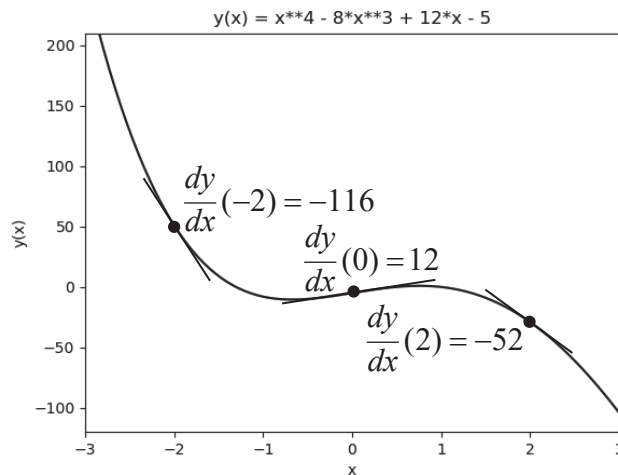
Find the derivative of $y = f(x) = x^4 - 8x^3 + 12x - 5$ wrt x . Then determine their values at $x = -2, 0$ and 2 .

DeepSeek will display the step-by-step process for finding the derivative of this function, along with the values of the derivatives at $x = -2, 0$, and 2 , then summarize the results on the screen as follows:

Summary of Results:

- Derivative: $f'(x) = 4x^3 - 24x^2 + 12$
- At $x = -2$: $f'(-2) = -116$
- At $x = 0$: $f'(0) = 12$
- At $x = 2$: $f'(2) = -52$

The derivatives calculated at these various positions represent the slopes, as shown in the figure.



Example Find the symbolic derivative of,

$$y = f(x) = \left[x + (x + \sin^2 x)^3 \right]^4$$

We can type the following command into DeepSeek to find the derivative of the function y as follow,

Find the derivative of $y = f(x) = (x+(x+(\sin(x))^2)^3)^4$ wrt x .

DeepSeek displays the detailed step-by-step process for finding the derivative of the function y (the on-screen details are omitted here). The result is,

$$\frac{dy}{dx} = 4 \left(x + (x + \sin^2(x))^3 \right)^3 \left[1 + 3 \left(x + \sin^2(x) \right)^2 (1 + 2 \sin(x) \cos(x)) \right]$$

To understand the process of finding derivatives in more complex cases like this, learners should carefully study the step-by-step solution displayed on the screen. This same process can also be used to find the n -order derivative of a given function, as shown in the following examples.

Example Find the second-order derivative of the function,

$$y = f(x) = 2x^3 + 7x^2 - 3x + 5$$

We can type the following command into DeepSeek to find the second-order derivative of the function y as follow:

Find the second derivative of $y = f(x) = 2*x^3+7*x^2-3*x+5$ wrt x .

DeepSeek will compute the second derivative and then display the result on the screen as,

$$f''(x) = 12x + 14$$

The examples shown above clearly demonstrate the usefulness and capability of DeepSeek in finding the derivatives of any given function. Moreover, if the function $y = f(x)$ depends on multiple variables, we can still use natural language commands to find the derivatives with respect to the appropriate variables as presented in the following examples.

Example Given the function,

$$z = f(x, y) = x e^{-(x^2 + y^2)}$$

Find the derivative of z with respect to x and y , respectively.

First, we can use DeepSeek to generate Python code to plot the distribution of the function $z = f(x, y)$ by typing: