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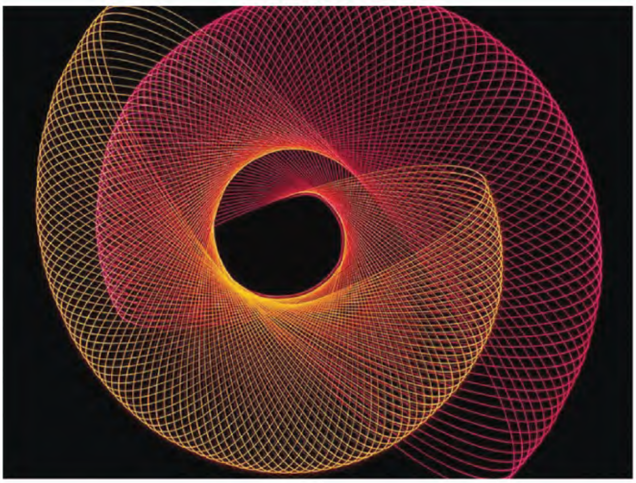
Calculus & Differential Equations with MATLAB



PRAMOTE DECHAUMPHAI

Jewels of Chula: In Commemoration of Chulalongkorn University's centennial celebrations

Calculus and Differential Equations with MATLAB

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Pramote Dechaumphai

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Message from the President

Chulalongkorn University, Thailand's first institution of higher education, took seed from the initiative of King Chulalongkorn, who wished to provide the opportunity of a university education for Thai people. Since its inception, the University has produced knowledgeable and morally conscious graduates, who have been able to pursue innovations that are beneficial to the country. It has served as a source of knowledge and the centre for the nation's academics motivating the country's development and ensuring its sustainable progress. It has always been the reliable source for Thai society.

In 2017, on the occasion of the centenary of its establishment, it is appropriate for Chulalongkorn University to publicize its successes in book form, with the title *The Project of Jewels of Chula: In Commemoration of Chulalongkorn University's Gems in Celebrating of the Establishment of the University*. This book is the result of the concerted effort and determination of faculty members and staff who have pooled their academic and research works, in a variety of academic disciplines, to provide essential knowledge that can be accessed by all of Thai society and the international communities.

The members of the Chulalongkorn Community are ready and proud to serve Thai society in order to continue its founder King Vajiravudh's intention to establish "the University of the Nation".

Professor Bundhit Eua-arporn, Ph.D.

President of Chulalongkorn University

Preface

MATLAB is a software widely used to solve mathematical problems that arise in the fields of science and engineering. These problems require profound understanding in the fundamentals of calculus and differential equations. MATLAB can solve many types of calculus problems and differential equations symbolically for exact closed-form expressions. If their exact solutions are not available, approximate solutions are obtained by using numerical methods.

This book, *Calculus and Differential Equations with MATLAB*, explains how to use MATLAB to solve calculus and differential equation problems in a clear and easy-to-understand manner. Essential topics in the calculus and differential equation courses are selected and presented. These topics include: limit, differentiation, integration, series, special functions, Laplace and Fourier transforms, ordinary and partial differential equations. Numerous examples are used to present detailed derivation for their solutions. These solutions are carried out by hands as normally done in classes and verified by using the MATLAB commands. Students thus understand detailed mathematical process for obtaining solutions. They, at the same time, realize the software capability that can provide the same solutions effectively within a short time. The solutions are then plotted to provide clear understanding of their behaviors.

The author would like to thank Miss Kaiumporn Phong-khachorn for her fine typing and his graduate students for proof-reading the manuscript. He would like to thank Chulalongkorn University Press and Alpha Science International for printing and distributing this book. He appreciates his wife Mrs. Yupa Dechaumphai for her support while writing this book.

Pramote Dechaumphai

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Chapter 1

Symbolic Mathematics by MATLAB

1.1 Introduction

Calculus and differential equations are requirement courses for science and engineering students. However, some students may not realize the importance of these subjects and simply take them for fulfilling their degree requirement. Such subjects, in fact, are essential because they are basis toward studying higher level courses so that more realistic problems can be solved. Most of the commercial software for design and analyzing scientific and engineering problems today were developed based on the knowledge of mathematics and computational methods. Good background in calculus and differential equations is thus needed prior to using the high-level commercial software correctly.

As scientists or engineers, solutions obtained from solving mathematical problems must be further interpreted so that their physical meanings are understood. They prefer to obtain solutions without spending a lot of time deriving them. There are many software today that can provide solutions to a large class of mathematical problems. These software can be used for finding roots of algebraic equations, taking derivatives and integrating functions, including solving for solutions of many differential equations. As a simple example, the software can perform integration,

$$\int_0^3 x^2 \, dx$$

numerically and returns the result of 9 immediately. At the same time, if preferred, the software can provide symbolic answer, such as,

$$\int x^2 \, dx = \frac{x^3}{3}$$

The software capability thus helps learning calculus considerably.

The idea for developing symbolic mathematics software started in early 1970's when a group of researchers at Massachusetts Institute of Technology developed a software so called MACSYMA (Mac Symbolic Manipulation Program). It was quite astonishing at that time because the solutions can be shown in the form of symbolic expressions instead of numbers. Lately, many symbolic manipulation capabilities of the software have been improved and can be used to ease learning calculus and differential equations.

1.2 Symbolic Mathematics Software

In the past, students need to memorize formulas when learning calculus. Basic formulas are required for finding derivative or integral of a given function. Proper steps must be taken carefully so that the final solutions are derived correctly. Few examples for finding solutions after taking derivative and performing integration of some functions are highlighted below.

Example Find the second-order derivative of the function,

$$f = \frac{x}{3 - 4\cos x}$$

To determine the second-order derivative of the given function above, the standard first-derivative formula is applied twice. The final solution is relatively lengthy as,

$$\frac{d^2 f}{dx^2} = \frac{16x\cos^2 x + 4\cos x(3x - 8\sin x) - 32x + 24\sin x}{(4\cos x - 3)^3}$$

Deriving for the above solution by hands, mistake may occur. With the help of symbolic computer software, the solution can be obtained instantly without any error. In addition, if the tenth or other higher-order derivatives of the above function are needed, the software can provide correct solutions in a very short time as well.

Example Find the first- and second-order derivatives of a more complex function,

$$g = \frac{x\tan x - 3\tan x - 21x^3 + 7x^4}{x^3 - 3x^2 + 6x - 18}$$

Again, the symbolic computer software can be used to provide the first-order derivative as,

$$\frac{dg}{dx} = -\frac{2x\tan x + 504}{(x^2 + 6)^2} + \frac{\tan^2 x + 43}{x^2 + 6} + 7$$

and then the second-order derivative as,

$$\frac{d^2 g}{dx^2} = \frac{2016x - 48\tan x}{(x^2 + 6)^3} - \frac{4x\tan^2 x - 6\tan x + 88x}{(x^2 + 6)^2} + \frac{2\tan x\,(\tan^2 x + 1)}{x^2 + 6}$$

Derivation of these solutions by hands would take a long time and is likely to contain error.

Example Integration is another topic learned in calculus course that many students do not appreciate. This is because they have to memorize many formulas and do not know when it will be used for solving realistic problems. Examples of integration learned in calculus course include indefinite integral, such as,

$$\int \frac{dx}{x^2+9} \quad = \quad \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$$

Also the definite integral, such as,

$$\int_1^2 \frac{x\,dx}{(x+1)(2x-1)} \quad = \quad \frac{1}{2}\ln(3) - \frac{1}{3}\ln(2)$$

Students can obtain solutions above in a short time if they use symbolic computer software.

Example Symbolic computer software can help us to solve some other types of problems that require a long time to do by hands. For example, it can be used to factorize the function,

$$h \quad = \quad x^4 + 26x^3 - 212x^2 - 1578x + 5859$$

to give,

$$h \quad = \quad (x+7)(x-9)(x-3)(x+31)$$

within a second.

Example Solving differential equations is another topic that most students do not like. This is because they are many approaches to follow depending on the types of differential equations. As an example, a general solution of the first-order ordinary differential equation,

$$\frac{dy}{dx} + y \quad = \quad 5$$

is,

$$y(x) \quad = \quad Ce^{-x} + 5$$

where C is a constant that can be determined from the given initial condition of the problem.

Some differential equations are more complicated, such as,

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y \quad = \quad 0$$

A general solution for the second-order ordinary differential equation above is,

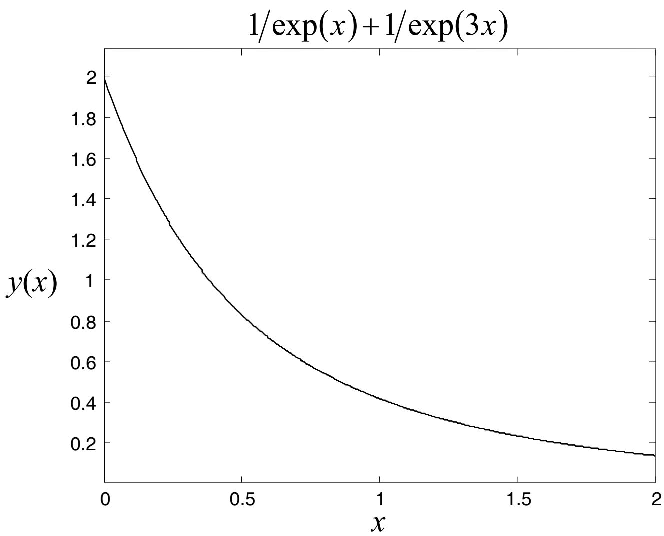
$$y(x) \quad = \quad C_1 e^{-x} + C_2 e^{-3x}$$

where C_1 and C_2 are constants that can be determined from the boundary conditions of the problem.

With the symbolic computer software, the above solutions can be obtained instantly. The software can also plot the solution behavior so that students understand its physical meaning clearly. For example, if the constants obtained after applying the boundary conditions are $C_1 = C_2 = 1$, the exact solution of the problem is,

$$y(x) \quad = \quad e^{-x} + e^{-3x}$$

The distribution of y that varies with x is plotted as shown in the figure.



1.3 History and Capability of MATLAB

MATLAB (MATrix LABoratory) was developed by Professor Cleve Moler, Head of Computer Science Department at the University of New Mexico in 1977. He wrote the LINPACK commands for solving algebraic equations and the EISPACK commands for analyzing eigenvalue problems, so that his students would not have to study FORTRAN language. Later, in 1983, Jack Little founded the Mathworks company to commercialize the software. The key capability of the software was to apply mathematical and computational methods through the use of matrices for solving academic problems. Soon, the software has received popularity mainly because of its ease of using.

In the past decade, MATLAB has included symbolic manipulation capability by linking its system operation with Maple and MuPad software. Such additional capability further increases the MATLAB popularity because a large class of mathematical problems can now be solved. The output solutions are in the forms of symbolic mathematical expressions instead of numbers. These solutions significantly help students in learning calculus and differential equation courses.

This book concentrates on how to use MATLAB to provide solutions in the forms of symbolic expressions similar to what we have learnt in classes. Selected topics which are important in calculus and differential equation courses are presented. Detailed derivations are illustrated prior to solving the same problems by using MATLAB commands. We will see that the same solutions are obtained instantly without any error from the software.

1.4 MATLAB Fundamentals

MATLAB is a huge computer software containing a large number of commands. Because this book concentrates on solving calculus problems and differential equations, only essential commands related to these topics are presented herein.

MATLAB assigns specific letters or names for some well-known quantities, such as,

- `i` and `j` denotes imaginary unit which is equal to $\sqrt{-1}$.
- `Inf` represents infinity which is $+\infty$, while $-\infty$ is denoted by `-Inf`.

- `NaN` refers to Not a Number, such as $0/0$ or `Inf/Inf`.
- `eps` is the acceptable tolerance which is 2.2204×10^{-16}.
- `pi` denotes π.

The value of π can be displayed up to n significant figures by using the command `vpa` (A, n) where A denotes the variable. As an example, the command for displaying the value of π with 200 significant figures is,

vpa

```
>> syms pi
>> vpa(pi,200)

ans =

3.141592653589793238462643383279502884197169399375
105820974944592307816406286208998628034825342117
0679821480865132823066470938446095505822317253594
0812848111745028410270193852110555964462294895493
0382
```

Similarly, $\sqrt{2}$ can be displayed with 200 significant figures by using the command,

```
>> vpa('(2)^(1/2)', 200)

ans =

1.4142135623730950488016887242096980785696718753 7
694807317667973799073247846210703885038753432764 1
5727350138462309122970249248360558507372126441214
9709993583141322266592750559275579995050115278206
05715
```

The command `syms` above is used to declare the specified variable as a symbol. For example, the variable x, y and t in the equation below,

$$u = 2x - 7y + t^2$$

can be declared as the three symbols by using the command,

syms

```
>> syms x y t
>> u = 2*x - 7*y + t^2
```

```
u =

t^2 + 2*x - 7*y
```

so that MATLAB won't expect the numerical values for them.

MATLAB contains several commands to manipulate and simplify algebraic expressions. These commands help reducing the complexity of the final symbolic expressions. Some useful commands are described herein.

The `collect` command expands the given expression and then collects similar terms together. For example,

$$f = (x+5)(x-3)$$

```
>> syms x
>> f = (x+5)*(x-3);
>> collect(f)

ans =

x^2 + 2*x - 15
```

collect

i.e., the final result is, $$f = x^2 + 2x - 15$$

The `expand` command expands and displays all the terms in the given function, e.g.,

$$g = \cos(x+y)$$

```
>> syms x y
>> g = cos(x+y);
>> expand(g)
ans =

cos(x)*cos(y) - sin(x)*sin(y)
```

expand

i.e., $$g = \cos x \cos y - \sin x \sin y$$

The `factor` command factorizes the given function to make it looks simpler. For example,

$$h = 6x^3 + 11x^2 - 16x - 21$$

```
>> h = 6*x^3 + 11*x^2 - 16*x - 21;
>> factor(h)
ans =
(3*x + 7)*(2*x - 3)*(x + 1)
```

factor

i.e., $$h = (3x+7)(2x-3)(x+1)$$

The `simplify` command simplifies the complex expression so that it is compact and easy to understand. As an example,

$$u = \frac{x+2y}{\dfrac{2}{x}+\dfrac{1}{y}}$$

```
>> u = (x + 2*y)/(2/x + 1/y);
>> simplify(u)
ans =
x*y
```

simplify

i.e., $$u = xy$$

The `simple` command is probably the most popular command because it combines the capability of the `collect`, `expand`, `factor` and `simplify` commands together. For example,

$$v = \frac{2x^5 - 5x^4 + 58x - 145}{2x - 5}$$

```
>> v = (2*x^5 - 5*x^4 + 58*x - 145)/(2*x - 5);
>> simple(v)
ans =
x^4 + 29
```

simple

i.e., the final solution is, $$v = x^4 + 29$$

It is noted that since the `simple` command contains several commands inside, detailed expressions during the simplification are appeared on the screen. These detailed expressions are omitted herein.

The `pretty` command is another useful command for transforming a symbolic expression into the rational form similar to those shown in textbooks. For example,

$$w = \frac{x \sin x}{x^2 + 12}$$

```
>> w = x*sin(x)/(x^2+12);
>> pretty(w)
```

pretty

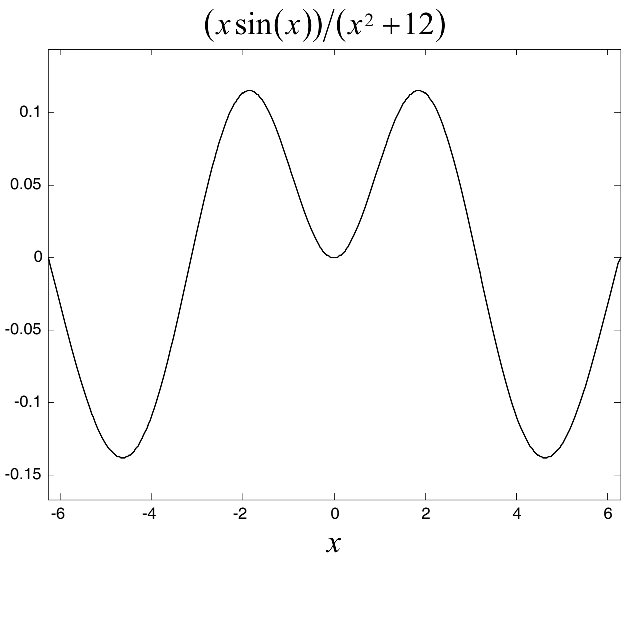
```
  x sin(x)
  --------
    2
   x  + 12
```

MATLAB contains the `ezplot` command that can be used to plot a given function effectively. As an example, if we would like to plot the *w* function above, we just simply enter the command,

```
>> ezplot(w)
```

ezplot

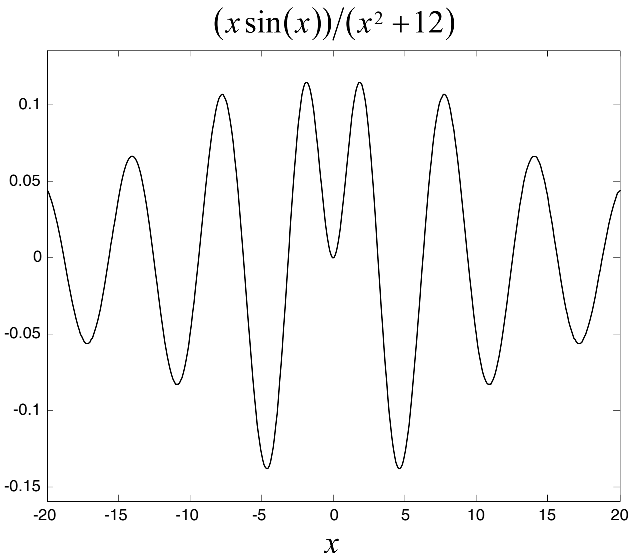
A plot of the *w* function will appear on the screen with the axis scaling adjusted automatically.



If we would like to plot the same w function within the interval of $-20 \le x \le 20$ along the x-axis, the above command is modified slightly to,

```
>> ezplot(w,[-20,20])
```

ezplot

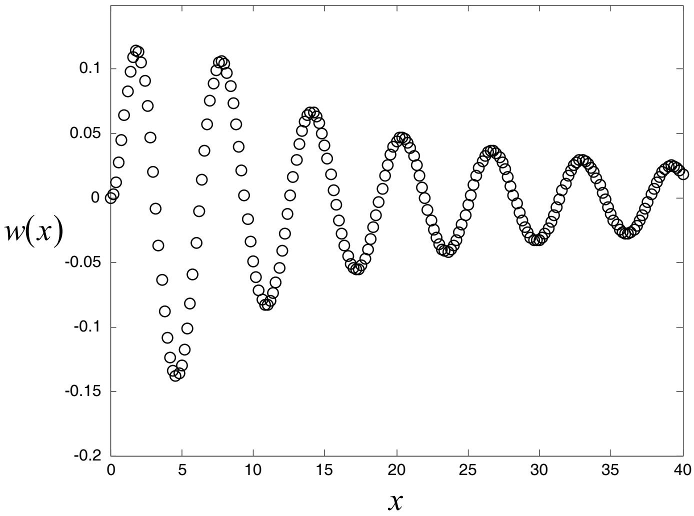


MATLAB contains the standard `plot` command that allows us to specify more details for plotting. For example, we want to compute the values of w function above at every x=0.2 for $0 \le x \le 40$. The values of w will be plotted as circle within the range of $-0.20 \le w \le 0.15$. The plot also includes labels on both horizontal and vertical axes as x and $w(x)$, respectively. In this case, the commands are as follows.

```
>> x = 0:.2:40;
>> w = x.*sin(x)./(x.^2+12);
>> plot(x,w,'ok')
>> axis([0,40,-.20,.15])
>> xlabel('x'), ylabel('w(x)')
```

plot

If preferred, we can include all the commands above in an m-file so that plotting details can be modified easily. In the argument of the `plot` command, ‘`o`’ denotes the circle while ‘`k`’ is for showing all circles in black. The plot generated from the commands is shown in the figure.



1.5 Concluding Remarks

In this chapter, the capability of symbolic manipulation software was introduced. The software helps finding solutions of basic problems learned in calculus and differential equation courses. These include: finding derivatives of functions, perform both definite and indefinite integrations, as well as solving for exact solutions of some differential equations. At present, there are many symbolic manipulation software suitable for learning and using in research work. Among them, MATLAB has received popularity due to its capability and ease of using.

Development history and essential features of MATLAB were briefly described. Few important commands were explained by using examples. These commands can help manipulating complex expressions and reduce them into simple forms. The solutions are plotted by using easy commands so that users can understand their physical meanings quickly. The chapter demonstrates advantages of using the symbolic manipulation software that can significantly reduce the effort for solving basic mathematical problems. We will appreciate these advantages in more details when we study essential calculus and differential equation topics in the following chapters.

Exercises

1. Study the symbolic manipulation capability in MATLAB by entering the command,

```
>> doc symbolic
```

   Then, make conclusions on how to:

   (a) simplify expressions
   (b) plot symbolic functions

   by setting up examples.

2. Use the command `collect`, `expand`, `factor`, `simplify` or `simple` to reduce or determine the following quantities,

   (a) $(-3)^4$ (b) $16^{-3/4}$
   (c) $3(x+6)+4(2x-7)$ (d) $(x-3)(4x+1)$
   (e) $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$ (f) $3x^{3/2}-9x^{1/2}+6x^{-1/2}$

3. Use the command `simple` to simplify the following functions,

   (a) $\dfrac{x^2+3x+2}{x^2-x-2}$ (b) $\dfrac{2x^2-x-1}{x^2-9}\cdot\dfrac{x+3}{2x+1}$

   (c) $\dfrac{x^2}{x^2-4}-\dfrac{x+1}{x-2}$ (d) $\dfrac{\dfrac{y}{x}-\dfrac{x}{y}}{\dfrac{1}{y}-\dfrac{1}{x}}$

   (e) $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2}$ (f) $\dfrac{(4x^3y^3)(3xy^2)^2}{\sqrt{xy}}$

4. Determine the product of the function f and g for each sub-problem. Then, employ the command `simple` to simplify their final expressions,

(a) $f = x^2 - 2x$; $g = (x+1)^2$

(b) $f = \sqrt{x-1}$; $g = \sqrt{x^2+2}$

(c) $f = \sin(x - x^2)$; $g = \cos(x^2 + x)$

(d) $f = e^{x-2}$; $g = \ln(x+3)$

(e) $f = \sin(x^2 - x)$; $g = \ln(x^2 + 2x)$

(f) $f = e^x$; $g = e^{-2x^2}$

5. Use the command `collect`, `expand`, `factor`, `simplify` or `simple` to yield simplest expressions of the following functions,

(a) $f = 6x^4 + 28x^3 - 7x^2 + 14x - 5$

(b) $g = (\cos x - \sin x)(\cos x + \sin x)(e^x + \sin x)(3x - 7)$

(c) $h = \dfrac{x\tan x - 3\tan x - 21x^3 + 7x^4}{x^3 - 3x^2 + 6x - 18}$

(d) $u = 3x + x^2 + 4\sqrt{x} + 2x\sqrt{x} + 2$

(e) $v = \dfrac{\sin x\tan^2 x + 6\sin x\tan x + 9\sin x}{x\cos^2 x - x - 7\cos^2 x + 7}$

(f) $w = \dfrac{15x + 3xy^2 + 5y^2 + y^4}{x^2(\cos x\sin y - \cos y\sin x) - \cos x\sin y - \cos y\sin x}$

6. Use the command `collect`, `expand`, `factor`, `simplify` or `simple` to prove that,

(a) $x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$

(b) $\sin(3x) = 3\sin x - 4\sin^3 x$

(c) $\tan(4x) = \dfrac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x}$

(d) $\cos^4 x = \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)$

(e) $(e^x + e^{-x})\tanh x = e^x - e^{-x}$

7. Use the command `collect`, `expand`, `factor`, `simplify` or `simple` to show that,

(a) $\tanh^2 x + \operatorname{sech}^2 x = 1$

(b) $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

(c) $\cosh\dfrac{x}{2} = \sqrt{\dfrac{\cosh x + 1}{2}}$

(d) $\cosh(4x) = 8\cosh^4 x - 8\cosh^2 x + 1$

(e) $\sinh^3 x = \frac{1}{4}\sinh(3x) - \frac{3}{4}\sinh x$

8. Use the `vpa` command to calculate and display the roots of the following prime numbers with 100 significant figures,

(a) $\sqrt{7}$ (b) $(157)^{2/3}$

(c) $(229)^{5/7}$ (d) $(443)^{8/9}$

(e) $(587)^{1/13}$ (f) $(881)^{27/31}$

9. Use the `ezplot` command to plot the following functions,

(a) $f = 3x^5 - 25x^3 + 60x$

(b) $g = \dfrac{5x^2 + 8x - 3}{3x^2 + 2}$

(c) $h = \frac{3}{4}(x^2 - 1)^{2/3}$

(d) $u = \sin(\pi/x)$

(e) $v = (x-2)^4 (x+1)^3 (x-1)$

10. Use the `plot` command to display the functions in Problem 9 again by showing essential details of their variations.

11. Study capabilities of the Mathematica and Maple software. Then, highlight their unique features and compare capabilities with MATLAB for manipulating symbolic expressions.

12. Use the `factor` command to simplify the function,

$$p = 3x^5 - 20x^3$$

Then, employ the `plot` command to display the variation of this function for (a) $-4 \le x \le 4$; $-500 \le p \le 500$ and (b) $0 \le x \le 4$; $-80 \le p \le 80$.

13. Use the `simplify` command to simplify the function,

$$q = x^4 - 2x^2 - 3$$

Then, use the `ezplot` command to display its variation for $-3 \le x \le 3$. Plot this function again but by using the `plot` command for $-3 \le x \le 3$ and $-6 \le q \le 6$.

14. Use the `ezplot` command to plot the following functions,

(a) $f = x^3 - x + 1$

(b) $g = x^4 - 3x^2 + x$

(c) $h = 3x^5 - 25x^3 + 60x$

Then, use the `plot` command with appropriate scaling in both the horizontal and vertical directions to clearly show their variations.

15. Use the `ezplot` command to display the following trigonometric functions,

(a) $\cos(2x)$ (b) $\sin\left(x - \frac{\pi}{2}\right)$

(c) $\cos(x + \pi)$ (d) $\tan(x + \pi)$

(e) $\cos\left(\frac{x}{6}\right)$ (f) $\cos\left(\pi + \frac{x}{2}\right)$

Then, use the `plot` command with appropriate scaling in both the horizontal and vertical directions to show their variations.

16. Given the function,

$$f = \sqrt[3]{1-x^3}$$

Find a proper command in MATLAB for determining the expression for f^{-1}. Then, plot to compare the variations between f and f^{-1}.

17. Given the function,

$$g = x\cos\left(\frac{10}{x}\right)$$

Use the `ezplot` command to plot its variation within the interval of $-2 \le x \le 2$. Then, use the `plot` command again to show the variation for $-0.5 \le x \le 0.5$. Suggest on how to plot the function when x approaches zero so that the variation is shown clearly.

Chapter 2

Calculus

2.1 Introduction

Calculus is an essential subject in mathematics required for science and engineering students. It contains two main topics which are the differentiation and integration of functions. The former one is based on understanding the determination of limits. Often, many students do not enjoy studying these topics because they have to memorize formulas for deriving solutions. Some solutions require a long time to derive by employing specific techniques. Furthermore, most students do not appreciate learning these topics because they don't know when the solutions will be used for realistic problems.

With the capability of the symbolic manipulation software today, solutions to calculus problems can be obtained rapidly.

Students can compare solutions obtained from the software with those derived by hands. So they will have more time to understand the meanings of the solutions. This chapter shows standard techniques to derive the solutions before using MATLAB commands to confirm the validity of them. Several examples will be presented with detailed derivation for the solutions. These solutions will also be plotted to increase understanding of their phenomena.

2.2 Limits

Limit of a function $f(x)$ when x approaches a, is defined by,

$$L = \lim_{x \to a} f(x)$$

Example Given a function,

$$f(x) = x^2$$

Then, $$L = \lim_{x \to a} f(x) = \lim_{x \to a} x^2 = a^2$$

We can use the `limit` command in MATLAB to obtain the solution by entering,

```
>> syms x a
>> f = x^2;
>> limit(f,x,a)

ans =

a^2
```

limit

Example Given a function,

$$g(x) = \frac{x^2 - a^2}{x - a}$$

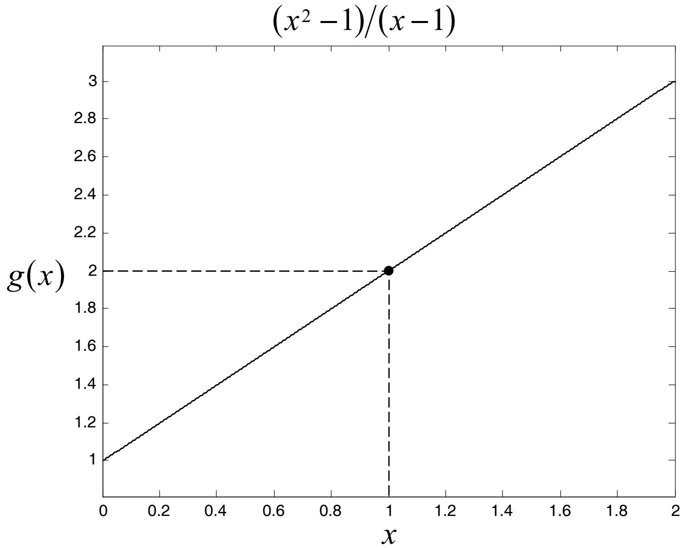
If we follow the simple procedure used above, we get,

$$L = \lim_{x \to a} g(x) = \lim_{x \to a} \frac{a^2 - a^2}{a - a} = \frac{0}{0}$$

The result cannot be determined and is not correct. To find the correct solution, we should observe the variation of this function $g(x)$ by plotting. If we assign the value of $a = 1$, then,

$$g(x) = \frac{x^2 - 1^2}{x - 1}$$

The plot of this function is shown in the figure. From the figure, the function $g(x)$ becomes 2 as x approaches 1.



The proper step to determine the limit of this problem is to first let $x = a + h$, where h is small. Then, substitute $x = a + h$ into the function $g(x)$ to get,

$$g(x) = \frac{(a+h)^2 - a^2}{a + h - a} = \frac{2ah + h^2}{h}$$

$$= 2a + h$$

Then, let h approaches zero, thus,

$$L = \lim_{x \to a} \left(\frac{x^2 - a^2}{x - a} \right) = 2a$$

The solution agrees with that shown in the graph at $a = 1$.

The same solution can be obtained instantly by using the `limit` command,

```
>> syms x a
>> g = (x^2-a^2)/(x-a);
>> limit(g,x,a)

ans =

2*a
```

limit

Finding limits of functions may require different methods depending on the function types. The examples below show standard techniques to determine limits for different types of functions.

Example Determine the limit,

$$L = \lim_{x \to 5} \frac{x^2 - 25}{x - 5}$$

This example is similar to the preceding example. The factoring technique can be used to find the solution as follows,

$$\begin{aligned} L &= \lim_{x \to 5} \frac{(x-5)(x+5)}{x-5} &&= \lim_{x \to 5} (x+5) \\ &= 5 + 5 &&= 10 \end{aligned}$$

Again, the `limit` command is employed to obtain the same solution,

```
>> limit('(x^2-25)/(x-5)',x,5)

ans =

10
```

limit

Example Determine the limit,

$$L = \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$$

If we simply substitute $x = 4$, we get $0/0$, which cannot be determined. A technique to find the limit of such function is to multiply its numerator and denominator by the conjugate value of the numerator, $\sqrt{x} + 2$, before taking the limit. Detailed derivation is as follows,

$$
\begin{aligned}
L &= \lim_{x \to 4} \frac{\left(\sqrt{x}-2\right)}{(x-4)} \cdot \frac{\left(\sqrt{x}+2\right)}{\left(\sqrt{x}+2\right)} \\
&= \lim_{x \to 4} \frac{(x-4)}{(x-4)\left(\sqrt{x}+2\right)} \qquad = \lim_{x \to 4} \frac{1}{\sqrt{x}+2} \\
&= \frac{1}{\sqrt{4}+2} \qquad = \frac{1}{4}
\end{aligned}
$$

Similarly, we can use the `limit` command to find such solution by entering,

```
>> limit('(sqrt(x)-2)/(x-4)',x,4)

ans =

1/4
```

<u>Example</u> Determine the limit,

$$
L = \lim_{x \to 0} \frac{\dfrac{1}{x+4} - \dfrac{1}{4}}{x}
$$

Again, if we substitute $x = 0$ directly into the function, we get $0/0$. The technique of multiplying by the conjugate value as shown in the preceding example is not applicable. A different technique of multiplying the numerator and denominator by an appropriate function is needed. For this particular problem, the appropriate function is $4(x+4)$ and the detailed procedure is as follows,

$$
\begin{aligned}
L &= \lim_{x \to 0} \frac{\left(\dfrac{1}{x+4} - \dfrac{1}{4}\right)}{x} \cdot \frac{4(x+4)}{4(x+4)} \\
&= \lim_{x \to 0} \frac{4-(x+4)}{4x(x+4)} \qquad = \lim_{x \to 0} \frac{-x}{4x(x+4)} \\
&= \lim_{x \to 0} \frac{-1}{4(x+4)} \qquad = \frac{-1}{4(0+4)} \\
&= -\frac{1}{16}
\end{aligned}
$$

The same solution is obtained by using the `limit` command as,

```
>> limit('(1/(x+4)-1/4)/x',x,0)
```

limit

```
ans =

-1/16
```

If the given function is more complex, the same `limit` command still provides solution immediately as demonstrated by the following examples.

<u>Example</u> Determine the limit,

$$L = \lim_{x \to -4} \frac{4-x}{5-\sqrt{x^2-9}}$$

The solution is obtained by typing the `limit` command followed by the given function, variable and limit value as,

```
>> limit('(4-x)/(5-sqrt(x^2-9))',x,-4)

ans =

-8/(7^(1/2) - 5)
```

which leads to the solution,

$$\lim_{x \to -4} \frac{4-x}{5-\sqrt{x^2-9}} = \frac{8}{5-\sqrt{7}} = 3.3981$$

<u>Example</u> Determine the limit,

$$L = \lim_{x \to 0} \frac{e^{x^3}-1}{1-\cos\sqrt{x-\sin x}}$$

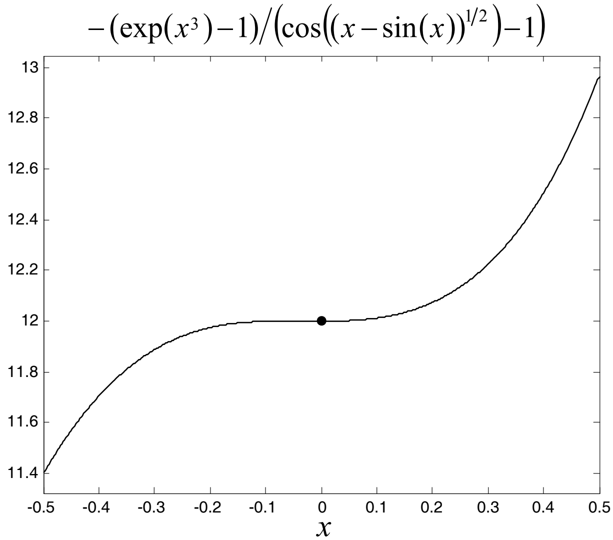
```
>> f = (exp(x^3)-1)/(1-cos(sqrt(x-sin(x))));
>> limit(f,x,0)

ans =

12
```

i.e., the solution is, $\lim_{x \to 0} \frac{e^{x^3}-1}{1-\cos\sqrt{x-\sin x}} = 12$

The solution can be verified by plotting such function through the use of the `ezplot` command as,

```
>> ezplot(f, [-.5,.5])
```

ezplot



<u>Example</u> Determine the limit of the function below as x approaches infinity,

$$L = \lim_{x \to \infty} \frac{9x+4}{\sqrt{3x^2-5}}$$

```
>> limit('(9*x +4)/(sqrt(3*x^2-5))',x,Inf)
ans =
3*3^(1/2)
>> double(ans)
ans =
    5.1962
```

double

i.e.,
$$\lim_{x \to \infty} \frac{9x+4}{\sqrt{3x^2-5}} = 3\sqrt{3} = 5.1962$$

<u>Example</u> The `limit` command can also be used to find the solution when the function contains two variables,

$$L = \lim_{\substack{x\to -1 \\ y\to 2}} \frac{x^2y + xy^3}{(x+y)^3}$$

```
>> syms x y
>> f = (x^2*y+x*y^3)/(x+y)^3;
>> L = limit(limit(f,x,-1),y,2)
```

limit

```
L =

-6
```

i.e.,
$$\lim_{\substack{x\to -1 \\ y\to 2}} \frac{x^2y + xy^3}{(x+y)^3} = -6$$

2.3 Differentiation

Differentiation is one of the most important topics in calculus. This is because the physics of most science and engineering problems are described by differential equations. Differential equations contain terms that are derivatives of the unknown variables. Finding for these unknown variables is the main objective for solving the differential equations. Thus, knowing how the derivatives of a function can be found is the first step toward learning the differential equations.

To find derivatives of a function as learned in classes, we need to apply basic formulas. Some formulas are easy to memorize while many others are difficult to recall. Furthermore, taking derivative of a complex function consumes a large amount of time and is likely to produce error.

Before using MATLAB command to find any derivative of a function, we start from understanding definition of the derivative. The derivative of a function $y(x)$ with respect to x is given by,

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Example Find the derivative of $y = f(x) = x^2$.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} 2x + \Delta x\end{aligned}$$

As $\Delta x \to 0$, then,

$$\frac{dy}{dx} = 2x + 0$$

or,

$$\frac{dy}{dx} = 2x$$

MATLAB contains the `diff` command that can be used to find the derivative effectively. In this example, the entered commands and solution are,

```
>> syms x
>> y = x^2;
>> diff(y,x)

ans =

2*x
```

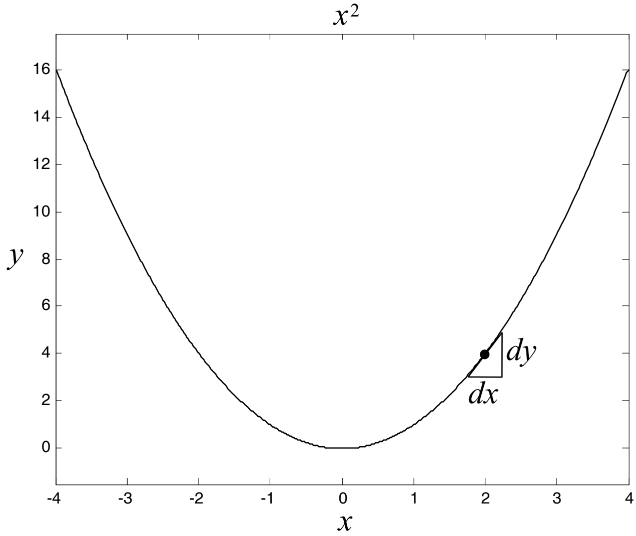
diff

Variation of the function $y = x^2$ can also be plotted easily as shown in the figure by using the command,

```
>> ezplot(y,[-4,4])
```

ezplot

The derivative dy/dx represents the slope at any x location. For example, at $x = 2$, the derivative $dy/dx = 2(2) = 4$.



<u>Example</u> Find the derivative of $y = f(x) = 2x - 3x^2$.

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2(x+\Delta x) - 3(x+\Delta x)^2 - (2x - 3x^2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2\Delta x - 6x\Delta x - 3(\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2 - 6x - 3\Delta x \\
&= 2 - 6x - 3(0) \\
\frac{dy}{dx} &= 2 - 6x
\end{aligned}
$$

Again, we can use the `diff` command to find the derivative of the function $y = f(x) = 2x - 3x^2$ as follows,

```
>> syms x
>> y = 2*x-3*x^2;
>> diff(y,x)

ans =

2-6*x
```

diff

Example Find the derivative of a constant function $y = f(x) = 5$.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{5-5}{\Delta x} = \lim_{\Delta x \to 0} 0 \\ &= 0\end{aligned}$$

That is, the derivative of a constant is zero.

```
>> y = 5;
>> diff(y,x)
```

diff

```
ans =

0
```

Example Find the derivative of the function $y = f(x) = \sqrt{x}$ for $x > 0$.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \to 0} \frac{x+\Delta x - x}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)} \\ &= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

We can use the `diff` command to obtain such solution directly,

```
>> syms x
>> y = sqrt(x);
>> diff(y,x)

ans =

1/(2*x^(1/2))
```

<u>Example</u> Find the derivative of the function $y = f(x) = \dfrac{x}{2x+3}$.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{x+\Delta x}{2(x+\Delta x)+3} - \dfrac{x}{2x+3}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{x+\Delta x}{2(x+\Delta x)+3} \cdot \dfrac{2x+3}{2x+3} - \dfrac{x}{2x+3} \cdot \dfrac{2(x+\Delta x)+3}{2(x+\Delta x)+3}}{\Delta x}$$

After simplifying it, we obtain,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{3}{(2x+2\Delta x+3)(2x+3)}$$

$$= \frac{3}{(2x+0+3)(2x+3)}$$

or,

$$\frac{dy}{dx} = \frac{3}{(2x+3)^2}$$

If we use the `diff` command, we can get the same result instantly,

```
>> syms x
>> y = x/(2*x+3);
>> diff(y,x)

ans =

3/(2*x + 3)^2
```

diff

In general, the given function $y = f(x)$ is complicated. Finding its derivative by hands consumes a large amount of time and may produce error. The `diff` command can eliminate such difficulty as shown in the following examples.

<u>Example</u> Find the derivative of the function,

$$y = f(x) = x^4 - 8x^3 + 12x - 5$$

Then, evaluate the result numerically at $x = -2, 0$ and 2.

Again, we can employ the `diff` command to find the derivative as follows,

```
>> syms x
>> y = x^4 - 8*x^3 + 12*x - 5;
>> dydx = diff(y,x)

dydx =

4*x^3 - 24*x^2 + 12
```

i.e.,

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 12$$

The `subs` command can then be used to compute the numerical results at $x = -2, 0$ and 2 as follows,

```
>> subs(dydx,-2)

ans =

  -116

>> subs(dydx,0)

ans =

    12

>> subs(dydx,2)

ans =

   -52
```

subs

These computed derivatives represent the slopes at $x = -2, 0$ and 2 as shown in the figure.