KKU's Textbook Fund

Mathematical Analysis I

Satit Saejung

Mathematical Analysis I

Prof. Satit SaeJung

Faculty of Science,

Khon Kaen Unversity

KKU's Textbook Fund

No. 178

Khon Kaen University, 2021

ISBN 978-616-438-598-6

Satit Saejung.

Mathematical Analysis I / Satit Saejung. -- 1st ed. – Khon Kaen : Department of Mathematics, Faculty of Science, Khon Kaen University, 2021.

94 p. : ill

1. Mathematical analysis. 2. Numbers, Real. 3. Functions, Continuous. 4. Differentiable functions. (1) Khon Kaen University. Faculty of Science. Department of Mathematics. (2) Title.

QA300 S253 v.1

ISBN 978-616-438-598-6

1st ed.

100 copies

Published by Learning and Teaching Innovation Center, Khon Kaen University

Printing House of Khon Kaen University

Price 260 bath

Preface

> Mathematical analysis is a branch of mathematics that includes the theories of differentiation, integration, measure, limits, infinite series, and analytic functions. These theories are usually studied in the context of real and complex numbers and functions. Analysis evolved from calculus, which involves the elementary concepts and techniques of analysis. Analysis may be distinguished from geometry; however, it can be applied to any space of mathematical objects that has a definition of nearness (a topological space) or specific distances between objects (a metric space).
>
> http://en.wikipedia.org/wiki/Mathematical_analysis

This subject is often referred to as "rigorous" calculus. We present and review many topics which have been studied for freshmen. But in this time, all the concepts are started with the precise definitions and followed by the corresponding theorems with proofs. Some illustrated examples are given. In this book, we use the rather standard symbols: $A \implies B$ and $A \iff B$ for the statements "If A, then B" and "A if and only if B", respectively.

This book is organized as follows. We first recall the real number system with some basic properties in Chapter 1. The Axiom of Completeness is a key property of this subject. Chapter 2 deals with a sequence of real numbers. The definition of a convergent sequence is given together with several important properties. Two interesting theorems, namely the Monotone Convergence Theorem and the Bolzano–Weierstrass Theorem are discussed. A criterion for the test of convergence of a sequence of real numbers is given in terms of being a Cauchy sequence. At the end of this chapter, we introduce the concept of an infinite limit of a sequence. The concept of continuous functions is given in Chapter 3. We show that the continuity of a function is closely related to the convergence of a sequence and hence many properties of continuous functions are deduced. We also discuss the definition of a limit of functions and an equivalent formulation in terms of a convergent sequence. Chapter 4 focuses on the differentiable functions. The Mean Value Theorem is given and applied to prove the well known L'Hôpital's Rule. The integrable functions are studied in Chapter 5. We recall and prove two fundamental theorems of calculus. Some properties about integrable functions are discussed.

Many exercises given at the end of each chapter are interesting and some of them will be used in the other chapter. Most of exercises are not difficult. However, some of them are tricky.

Finally, I would like to thank all the students, especially Jedsada Senasukh, in my class for their comments and suggestions on this book.

Satit Saejung

January, 2021

Contents

1

The real number system

In this book, we let

$$\begin{aligned}\mathbb{R} &:= \{\text{real numbers}\}\\ \mathbb{Q} &:= \{\text{rationals}\}\\ \mathbb{Q}' &:= \{\text{irrationals}\}\\ \mathbb{Z} &:= \{\text{integers}\}\\ \mathbb{N} &:= \{\text{natural numbers}\}.\end{aligned}$$

Hence, we have the following inclusions:

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \quad \text{and} \quad \mathbb{Q}' \subsetneq \mathbb{R}.$$

In this chapter, we review some basic properties of real numbers and of two usual binary operations—addition and multiplication. We use the absolute value as a tool to measure distance between two real numbers. The Axiom of Completeness is the key property of this chapter. At the end of this chapter, we recall the concept of the extended real line.

1.1 Some basic properties

We assume that the readers are familiar with the following properties of the two binary operations, that is, the addition $(+)$ and the multiplication $(\cdot)$. (To keep the notation light, we sometimes write ab for $a \cdot b$.) For each $a, b, c \in \mathbb{R}$, we have the following statements:

Commutativity: $a + b = b + a$ and $ab = ba$;

Associativity: $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$;

Identity: $a + 0 = a$ and $a \cdot 1 = a$;

Inverse:
- There exists an element $-a$ such that $a + (-a) = 0$;
- If $a \neq 0$, there exists an element a^{-1} such that $aa^{-1} = 1$;

Distributivity: $a(b + c) = ab + ac$.

Remark 1.1. If $a, b, c \in \mathbb{R}$ and $a = b$, then $a + c = b + c$ and $ac = bc$.

Proposition 1.2 (Cancellation Law). *Let $a, b, c \in \mathbb{R}$.*

(a) *If $a + c = b + c$, then $a = b$.*

(b) *If $ac = bc$ and $c \neq 0$, then $a = b$.*

Proof. Let $a, b, c \in \mathbb{R}$.

(a) Assume that $a + c = b + c$. Then there is an element $-c$ such that $c + (-c) = 0$. This implies that $(a+c)+(-c) = a+(c+(-c)) = a+0 = a$. Similarly, $(b+c)+(-c) = b$. Since $(a + c) + (-c) = (b + c) + (-c)$, we have $a = b$.

(b) Assume that $ac = bc$ and $c \neq 0$. Then there is an element c^{-1} such that $c \cdot c^{-1} = 1$. This implies that $(ac) \cdot c^{-1} = a(cc^{-1}) = a \cdot 1 = a$. Similarly, $(bc) \cdot c^{-1} = b$. Since $(ac) \cdot c^{-1} = (bc) \cdot c^{-1}$, we have $a = b$. □

Corollary 1.3. *If $a \in \mathbb{R}$, then $a \cdot 0 = 0$.*

Proof. Let $a \in \mathbb{R}$. Note that $0+0 = 0$. It follows that $a\cdot0+a\cdot0 = a(0+0) = a\cdot0 = 0+a\cdot0$. By the Cancellation Law, we have $a \cdot 0 = 0$. □

Proposition 1.4. *Let $a, b \in \mathbb{R}$. The following statements are true:*

(a) $ab = 0 \iff a = 0$ *or* $b = 0$.

(b) $-(-a) = a$.

(c) $(-1)a = -a$.

(d) $(-a)(-b) = ab$.

(e) $(a^{-1})^{-1} = a$ *if* $a \neq 0$.

(f) $(ab)^{-1} = a^{-1}b^{-1}$ *if* $a \neq 0$ *and* $b \neq 0$.

Proof. Let $a, b \in \mathbb{R}$.

(a) ($\Rightarrow$) Assume that $ab = 0$. If $a = 0$, then we are done. We now assume that $a \neq 0$. Then there is an element a^{-1} such that $a^{-1}a = 1$. This implies that $b = 1 \cdot b = (a^{-1}a)b = a^{-1}(ab) = a^{-1} \cdot 0 = 0$.

($\Leftarrow$) Assume that $a = 0$ or $b = 0$. It follows from the preceding corollary that $ab = 0$.

(b) Note that $a + (-a) = 0$ and $(-a) + (-(-a)) = 0$. It follows that $a + (-a) = (-(-a)) + (-a)$. By the cancellation law, we have $a = -(-a)$.

(c) Note that $a + a \cdot (-1) = a \cdot 1 + a \cdot (-1) = a(1 + (-1)) = a \cdot 0 = 0$ and $a + (-a) = 0$. This implies that $a + a \cdot (-1) = a + (-a)$. By the cancellation law, we have $(-1)a = a \cdot (-1) = -a$.

(d) We use (b) and (c). Note that

$$(-a)(-b) = ((-1)a)(-b) = a((-1)(-b)) = a(-(-b)) = ab.$$

(e) Assume that $a \neq 0$. Then there is an element a^{-1} such that $a^{-1}a = 1$. This implies that $a^{-1} \neq 0$. Then there is an element $(a^{-1})^{-1}$ such that $(a^{-1})^{-1}a^{-1} = 1$. In particular, $a^{-1}a = (a^{-1})^{-1}a^{-1}$. By the cancellation law, we have $a = (a^{-1})^{-1}$.

(f) Assume that $a \neq 0$ and $b \neq 0$. It follows from (a) that $ab \neq 0$. Then there is an element $(ab)^{-1}$ such that $(ab)^{-1}(ab) = 1$. We note that $(a^{-1}b^{-1})(ab) = (a^{-1}b^{-1})(ba) = (a^{-1}(b^{-1}b))a = (a^{-1} \cdot 1)a = a^{-1}a = 1$. This implies that $(ab)^{-1}(ab) = (a^{-1}b^{-1})(ab)$. By the cancellation law and $ab \neq 0$, we have $(ab)^{-1} = a^{-1}b^{-1}$. □

We also write

$$a - b := a + (-b) \quad \text{and} \quad a/b := ab^{-1}$$

where the latter is defined for $b \neq 0$.

Corollary 1.5. *Let $a, b \in \mathbb{R}$. Then the following statements are true.*

(a) $-(a - b) = b - a$.

(b) *If $a \neq 0$ and $b \neq 0$, then $(a/b)^{-1} = b/a$.*

Proof. Let $a, b \in \mathbb{R}$.

(a) We note that $-(a - b) = (-1)(a + (-b)) = (-1)a + (-1)(-b) = -a + b = b - a$.

(b) We assume that $a \neq 0$ and $b \neq 0$. It follows that $(a/b)^{-1} = (ab^{-1})^{-1} = a^{-1}(b^{-1})^{-1} = a^{-1}b = b/a$. □

We assume the following axiom:

Positivity Axiom:

- If $a, b > 0$, then $a + b > 0$ and $ab > 0$.
- For each $a \in \mathbb{R}$, one of the following statements holds true:

$$a > 0 \quad \text{or} \quad a = 0 \quad \text{or} \quad -a > 0.$$

Now we define the following relation: for $a, b \in \mathbb{R}$

$$a < b \overset{\text{def}}{\iff} b - a > 0.$$

We also write $a \leq b$ for the situation $a < b$ or $a = b$. Moreover, we write $a < b < c$ for the situation $a < b$ and $b < c$. The expressions $a < b \leq c$, $a \leq b < c$, and $a \leq b \leq c$ can be interpreted similarly.