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# LOW–REYNOLDS–NUMBER FLUID DYNAMICS:

## Applications to Hindered Fluid and Particle Transport

Panadda Dechadilok

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APPLICATIONS TO HINDERED FLUID
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# PREFACE

The objective of this book is to provide the readers with a theoretical background involving dynamics of confined viscous flows as well as its effect on particle diffusion and convection based on my research experience for the last fourteen years. Fluid dynamics is a branch of mechanics concerned with the relationship between dynamic motions of flows of fluids, substances that continuously deform under the applied shear stress, and the presented forces. If the length scale of the system of interest is much larger than the size of the fluid molecules, the fluid is viewed from a macroscopic point of view and is considered to be a "continuum", a continuous mass rather than discrete molecules, and, if the fluid velocity is much lower than the speed of light, the study of behaviors of flows within such system is referred to as classical fluid dynamics, a subfield of continuum mechanics. Although not traditionally included in classical mechanics, one of the core courses for physics majors in all undergraduate and graduate-level curriculums, classical fluid dynamics is still important and useful as it has applications in many research areas in physics such as geophysics, plasma physics, and physics of biological systems in addition to various additional fields including biotechnology, biochemistry, and engineering.

Due to the fact that there exist a large number of books covering classical fluid dynamics, a question arises; how does this book differ from previously existing books? The first distinction is that the content of this book primarily focuses on dynamics of flows characterized by vanishingly small Reynolds numbers, commonly referred to as creeping flows or Stokes flows. A product of the system length and velocity scales divided by the fluid kinematic viscosity, the Reynolds number is an approximate indicator of the relative contribution of the inertial effect (or convection of momentum) to momentum transfer within the flow and that of the effect of viscous dissipation. A very small value of the Reynolds number indicates that the effect of viscous dissipation on fluid pressure and velocity is dominant whereas the inertial effect is very small. Example of low-Reynolds-number flows or Stokes flows include flows in systems with small length scales such as flows in microchannels as well as flow in systems with small velocity scale such as glacial flows. It has been long observed that Stokes flows display behaviors that differ greatly from those of high-Reynolds-number flows often observed in our daily lives, rendering the study of their motions an interesting topic on its own. Another unique point of this book is that, to accommodate a growing field of purification and separation technology utilizing confined Stokes flows, studies of hindered transport of fluids and particles confined in long channels based on my research expertise are presented as examples of the applications of low-Reynolds-number fluid dynamics in Chapter 5, 6 and 7.

As for the usage of this book, its main objective is to provide the necessary introductory theoretical background involving Stokes flows for readers who may not have a prior background in fluid mechanics but have a solid background in vector calculus and in solving differential equations. It is recommended that, to find the book comprehensible, a basic background including freshmen level physics, vector calculus and partial differential equations is required. Other mathematical concepts (such as a gradient of the fluid velocity which is a tensor) are explained in the book in the appendix. The brief overall introduction of Stokes flows is presented in Chapter 1. To make the book self-contained, Chapter 2 is devoted to the derivation of the Navier-Stokes equation as an analogy of Newton's second law in a form applicable to linear momentum transfer in fluid flows; effects of channel geometrics on confined unidirectional flows are presented as examples. This is followed by an introduction of the Reynolds number as a criterion in determining the basic behavior of the flows (based on scaling analysis) in Chapter 3. Properties of Stokes flows with all their peculiarities are discussed in detail in Chapter 4. Discussions involving hindered transport and electrokinetic phenomena based on my research experience are contained within the last three chapters. Effects of the hydrodynamic drag (altered by the presence of the confining channels) on diffusion and convection of particles immersed in Stokes flows, and the application involving size-based separation, are presented in Chapter 5. Electrokinetic effects on behaviors of confined electrolytic flows are discussed in Chapter 6, whereas the combined effects of particle-channel hydrodynamic, electrostatic and electrokinetic interactions on convection and diffusion of particles suspended in particulate flows confined in long channels are presented in Chapter 7. Chapter 8 contains an overall summary and a conclusion. In each chapter, the simplest systems consisting of confined particles (with the size much larger than the fluid molecules) in long channels with uniform cross-sections are considered, although the same theoretical treatment is applicable to systems with more complex geometries. This book is self-contained, and even though it covers a fraction of the field of fluid dynamics (namely, the dynamics of Stokes flows), I truly hope it would be useful, at least to some extent, for researchers, students and everyone interested in fluid mechanics and related phenomena.

The content of this book has undergone a peer review as a part of a publication process with the reviewers being experts from multiple institutes. I am very grateful to Chulalongkorn University Press and to the experts who are the anonymous reviewers for taking the time to review this book, making it more complete. I also would like to thank Professor William M. Deen, my advisor, for his kindness and insight that kindled my life long research interest in hydrodynamics and transport phenomena. I am also grateful to Professor Michael P. Brenner and Professor Howards A. Stone for giving me an enormous opportunity, initiating me into the field of fluid mechanics and providing me with a solid theoretical background, Professor Patrick S. Doyle for his very interesting lectures for the course 10.50 at MIT, and Dr. Yuttana Roongthumskul for his suggestions and inputs. I am also grateful to all the students enrolling in 2304520 over the years; their questions and inquiries have helped deepen my understanding of phenomena involving all kinds of fluid flows and not just Stokes flows. (One can say that they are my instructors.) Finally, I would like to thank my family (especially my parents) for being so understanding and supportive throughout my writing process, and for allowing me to allocate some of the time I should have spent with them and spend it in completing this book. They have done all of this with such good grace; I truly and deeply appreciate it.

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May 31st, 2020

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# CHAPTER 1

# INTRODUCTION

## 1.1 Introduction

Fluid mechanics is a field in continuum mechanics devoted to analyses of phenomena involving liquid and gaseous flows: substances of which the "relaxation time", the time needed for the materials to adjust to the applied shear stress, is very small compared to the "time of observation", the time scale of the experiment. In the field of rheology, the ratio between the relaxation time scale and the time scale of observation is referred to as the *Deborah number* (De). Solids are substances characterized by a large value of De, whereas, viscoelastic materials displaying properties of both solids and fluids, are characterized by De that is of order unity. For fluids, De << 1. For instance, the relaxation time of water is about $10^{-12}$ s, resulting in the Deborah number being much less than 1. This very fast "adjustment" or a very fast material deformation rate makes it impossible to quantify the "strain" corresponding to the shape change of such material; what the applied shear stress produces is a "strain rate" (the rate at which the material is deformed) that increases as a function of the magnitude of the applied shear stress. If the length scale of the considered system is much larger than the size of the fluid molecules, the "continuum assumption", the assumption that the fluids are continuum, is a good approximation. Flow velocity and fluid pressure are well-defined at infinitesimally small points and vary continuously. The relationship between the fluid macroscopic response to applied stress and its basic properties is the scope of the present study.

It has been observed that behaviors of most flows observed in our daily lives greatly differ from those of flows where viscous dissipation dominates the momentum transfer process. For instance, in every day's life, "mixing" is often seen as an irreversible process. If a small amount of milk is poured into a cup of coffee which is, then, stirred by a spoon in a clockwise direction, milk molecules are irreversibly mixed with the coffee in the cup, and "unmixing" is known to be impossible. Even if the spoon is, then, used in stirring the coffee in a counterclockwise direction, this cannot return all the diffusing milk molecules into their original locations. If the same process is repeated by using a more viscous fluid, however, "unmixing" becomes possible to achieve. As shown by G.I. Taylor in the experiment recorded in a film made for National Committee for Fluid Mechanic Films (1967), a color dye drop was gently placed in a glycerin contained between two concentric cylinders. The inner cylinder was, then, rotated slowly in a clockwise direction for a few rounds, and the color dyes were mixed with glycerin. An equal number of rounds of the inner cylinder rotation in a counterclockwise direction, however, reversed the mixed color dyes back to being contained within a single droplet (in an approximate sense) at its original location (Taylor, 1967; Dechadilok, 2017). The "unmixing" happening in a viscous fluid is, in G.I. Taylor's own words, one of the "surprising situations which almost made one believe that the fluid has memory of its own" (Taylor, 1967). As will be discussed further in Chapter 3 and 4, this is due to the time scale of viscous dissipation being generally very small, and time enters the problem only as a parameter; the fluid velocity and pressure respond spontaneously to temporal changes. The content of this book is devoted to such flows where viscous dissipation plays a prominent role, giving rise to several strange phenomena which can be utilized in many different fields.

## 1.2 Flow analysis and Reynolds number

Based on the discussion in Sec. 1.1, it is, therefore, not surprising that the most important parameter in the field of fluid dynamics is the dimensionless parameter that serves as a criterion in determining the relative importance of the effect of convection of momentum and that of the viscous dissipation during the process of momentum transfer within fluid flows known as the *Reynolds number* (Re). Defined as the product of the fluid density ($\rho$) and the length ($L$) and velocity scales ($U$) of the system divided by the fluid shear viscosity ($\mu$), Re can be viewed as the ratio between the estimated time scale for *viscous dissipation*, sometimes seen as "diffusion of momentum", $L^2/(\mu/\rho)$, and the estimated time scale of *linear momentum convection*, $L/U$, as will be discussed in detail in Chapter 3. If the value of Re characterizing the flow is high, momentum convection is much "faster" than viscous dissipation and is, therefore, a dominant factor in the momentum transfer process. Observed phenomena involving *high-Reynolds-number flows* are those often associated with liquid or gaseous flows observed in every day's life such as eddy formation and, if Re is high enough, turbulence (Dechadilok, 2017). Examples of high-Reynolds-number flows include liquid flows generated by swimming of organisms that can be observed by naked eyes, a smoke flowing from a chimney, an aerodynamic air flow past an airfoil and flows generated by bird locomotion.

If, instead, the Reynolds number characterizing the flow is very small, viscous dissipation is a much "faster" process, and its effect is more prominent. Such flows are referred to as *low-Reynolds-number flows*, and, as discussed in Sec. 1.1, their behaviors are known to differ from high-Reynolds number flows in numerous ways. Examples of these flows are glacial flows observed in the mountains (characterized by small velocity scales) and flows of viscous fluids such as those of honey or glycerin. These fluid flows are also referred to as *Stokes flows* (in remembrance of G. G. Stokes who pioneered the early theoretical study of low-Reynolds-number hydrodynamics) or *creeping flows* (as they are often associated with small velocity scale in addition to small length scale and high fluid viscosity).

## 1.3 Applications of low-Reynolds-number fluid dynamics

As aforementioned, this book focuses on Stokes flows or creeping flows observed in systems where either the length and fluid velocity scales are very small, or the fluid viscosity is very large. For instance, flows produced by sedimentation of small particles (Booth, 1954; Stigter, 1980) or microorganism swimming (Lauga and Power, 2009) are known to be Stokes flows. So are transcapillary fluid flows during the process of renal urine formation (Deen et al., 2001). The development of microfluidic devices during the past few decades renders on-chip chromatography and single biomolecule sensing possible; in many circumstances, hydrodynamic flows contained in the devices are characterized by low Reynolds numbers (Stone et al., 2004; Trombley and Ekiel-Jezewska, 2019) due to the small length scale of the devices. Manipulations of such flows require a basic knowledge of dynamics of low-Reynolds-number flows. Stokes flows characterized by low Reynolds numbers are, therefore, the primary focus of this book.

## 1.4 Conclusion

The objective of this book is to provide the readers with the basic theoretical background for analyses of Stokes flows with applications involving confined flows and hindered transport in long channels. Its content begins with the equation of motions, the Cauchy momentum equation and the continuity equation, before shifting the focus to flows of incompressible Newtonian fluids governed by the Navier-Stokes equation. The Reynolds number is introduced and the simplification based on its value is discussed in detail. A review of properties of flows characterized by low Reynolds numbers is given. The content involving my research expertise, hindered transport and ionic solution flows, are contained in the last three chapters. To be presented as an example, the considered system is one of the simplest systems, a long channel with the uniform cross-section containing a dilute particulate flow. As will be subsequently discussed, the Reynolds number characterizing the confined flow is generally very small. For an uncharged system, the increase in the particle size relative to the size of the channel cross-section leads to an increase drag exerted on the particle, causing changes in the particle diffusivity and convection rate. The hydrodynamic interaction between the suspended particles and the channel walls allows experimentalists to design and utilize microchannels as tools for sensing and separation.

In addition, charge effects on the dynamics of the confined flows and its applications are extensively reviewed. When a charged solid surface is in contact with an electrolyte solution, a layer of oppositely charged ions is formed in the vicinity of the solid-fluid interface. The combination of the counterion diffuse layer and the electrical charges on the solid surface is known as the electrical double layer; the characteristic length scale of the diffuse layer, referred to as the Debye length, is inversely dependent on the ionic concentration of the electrolyte solution. The effect of the Debye length on the electrolytic flows confined in long channels are reviewed. Finally, the combined effect of the particle size and charges on the particle fluxes are discussed, and the opposing effects of the particle-channel electrostatic interaction and the electrokinetic distortion of the double layer on confined particle diffusion and convection are examined.

# CHAPTER 2

# CAUCHY MOMENTUM EQUATION AND NAVIER-STOKES EQUATION: ANALYSIS OF FULLY DEVELOPED FLOWS IN LONG CHANNELS

## 2.1 Introduction

The objective of this book is to provide the readers with the theoretical background regarding fluid mechanics of low-Reynolds-number flows where the effect of viscous dissipation is the dominant factor in the momentum transfer process within the flow. As aforementioned, all the theoretical analyses in the present and subsequent chapters are for the systems with the characteristic length scales much larger than the size of the fluid molecules such that continuum mechanics is applicable. To make the book self-contained, this chapter is devoted to the derivation of the governing equations for fluid velocity and pressure with a primary objective of laying a groundwork for subsequent analyses of confined flows, beginning with the governing equations for flows of all fluid types, the Cauchy momentum equation (that is an analogy of Newton's second law) along with the continuity equation (based on the conservation of mass), in Sec. 2.2. The surface and body forces affecting the fluid motion are discussed in Sec. 2.3. Properties of an incompressible Newtonian fluid, and the governing equation for its velocity and pressure, the Navier-Stokes equation, are stated in Sec. 2.4 and 2.5, respectively. A pressure variation in a static fluid is discussed in Sec. 2.6, whereas the discussion about generalized Newtonian fluids is given in Sec. 2.7. Finally, the theoretical analysis of fully developed fluid flows confined in long channels are presented in Sec. 2.8.

## 2.2 Cauchy momentum equation: Newton's second law in the form applicable to momentum transport in fluid flows

By following a method introduced by Deen (1998), the derivation of the governing equations for fluid velocity and pressure in this book begins with one of the most well-known and utilized equation for physicists. According to Newton's second law, the change of the linear momentum of a solid body with a constant mass is related to the exerted net force as follows.

$$m\frac{d\mathbf{v}}{dt} = \sum_i \mathbf{F}_i \tag{2.1}$$

where $m$ is the object mass, and $\mathbf{v}$ is the object velocity. $\sum_i \mathbf{F}_i$ is the total force: the sum of all the forces exerted on the object. The objective of this section is to obtain Eq. (2.1) in the form applicable to fluid flows. If the change of the linear momentum in an arbitrary control volume ($V_M$) with the bounding surface ($S_M$) moving at the arbitrary velocity ($\mathbf{v_s}$) is considered, Eq. (2.1) can, then, be modified as

$$\frac{d}{dt}\left(\int_{V_M(t)} \rho\mathbf{v}dV\right) = \sum_i \mathbf{F}_i - \int_{S_M(t)} \mathbf{n}\cdot\rho\mathbf{v}\left(\mathbf{v}-\mathbf{v_s}\right)dS \tag{2.2}$$

where $\rho$ is the fluid density, and $\rho\mathbf{v}$ is the linear momentum per unit volume. The second term on the right-hand side is the convective loss of linear momentum through its *flux* across $S_M$. The change of linear momentum accumulated in $V_M$ are caused by two factors: the convective loss through $S_M$ and the sum of the exerted forces. According to the Leibniz rule[1] for differentiating the volume intergral, the left-hand side of Eq. (2.2) can be rewritten as shown below.

$$\frac{d}{dt}\left(\int_{V_M(t)} \rho\mathbf{v}dV\right) = \int_{V_M(t)} \frac{\partial(\rho\mathbf{v})}{\partial t}dV + \int_{S_M(t)} \mathbf{n}\cdot\mathbf{v_s}\left(\rho\mathbf{v}\right)dS \tag{2.3}$$

Applying the divergence theorem[2] to change the surface integral of the last terms on the right-hand side of Eqs. (2.2) and (2.3) into a volume integral and substituting the expression in Eq. (2.3) into Eq. (2.2), one obtains the following expression.

$$\int_{V_M(t)} \left[\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot\left(\rho\mathbf{v}\mathbf{v}\right)\right]dV = \sum_i \mathbf{F}_i \tag{2.4}$$

where $\rho\mathbf{v}\mathbf{v}$ is a tensor discussed in Sec. A3 in the appendix. The expression on the left-hand side of Eq. (2.4) can be rewritten (using the chain rule) as

[1] The Leibniz rule for differentiating integrals is discussed in Sec. A4 of the appendix.

[2] The discussion involving the divergence theorem for scalar functions, vectors and tensors is presented in Sec. A3 of the appendix.

$$\int_{V_M(t)} \left[ \frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) \right] dV = \int_{V_M(t)} \mathbf{v}\left[ \frac{\partial(\rho)}{\partial t} + \nabla\cdot(\rho\mathbf{v}) \right] + \rho\left[ \frac{\partial(\mathbf{v})}{\partial t} + \mathbf{v}\cdot\nabla(\mathbf{v}) \right] dV \tag{2.5}$$

If the bounding surface ($S_M$) of $V_M(t)$ is moving at the same velocity as the local fluid velocity, the total mass within $V_M(t)$ is constant. Conservation of mass leads to the expression as shown below (Batchelor, 1967).

$$\frac{d}{dt}\int_{V_M(t)} \rho dV = \int_{V_M(t)} \left[ \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) \right] dV = 0 \tag{2.6}$$

Substituting the expressions in Eqs. (2.5) and (2.6) into Eq. (2.4), one obtains the expression analogous to Newton's second law that is applicable to fluid flows as follows.

$$\frac{d}{dt}\left[ \int_{V_M(t)} \rho\mathbf{v} dV \right] = \int_{V_M(t)} \rho\frac{D\mathbf{v}}{Dt} dV = \sum_i \mathbf{F}_i \tag{2.7}$$

where

$$\rho\frac{D\mathbf{v}}{Dt} = \rho\left( \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} \right) \tag{2.8}$$

The differential operator in Eq. (2.8), "$D/Dt$", is referred to as the *material derivative* or the *substantial derivative*. $\rho(D\mathbf{v}/Dt)$ is the rate of change of the linear momentum as seen by the observer that is moving at the same velocity as that of the fluid: a combination of the linear momentum change due to the time dependence of the fluid velocity $(\rho\partial\mathbf{v}/\partial t)$ and the observed change of the linear momentum caused by the spatial variation of the fluid velocity and the movement of the observer $(\rho\mathbf{v}\cdot\nabla\mathbf{v})$. Equation (2.7), therefore, can be viewed as Eq. (2.1) or Newton's second law in the form that is applicable to fluid flows. What remains to be discussed is the net force $\left(\sum_i \mathbf{F}_i\right)$.

## 2.3 Stresses and body forces

The external forces contributing to the net force are often divided into two types: the forces that are exerted directly on the fluid mass referred to as *the body forces* and the forces that can be expressed in terms of the forces per unit area or *stresses*. Examples of the body forces include the *gravitational force* written in a volume integral form as shown below.

$$\mathbf{F}_g = \int_{V_M(t)} \rho\mathbf{g} dV \tag{2.9}$$

Under special circumstances, there are also contributions from other physical fields. For instance, if the fluid is an electrolyte solution and there is an applied electric field, the force that an electric field exerts on small ions in the solution is also a body force that must be included in equations of motion. (The contribution of the electrical body force to the change in fluid velocity is discussed in detail in Chapter 6.) In the present chapter, if **f** is the force per volume contributed by physical fields (such as an electric field or a magnetic field), the force such fields exert on the mass contained in $V_M$ is simply

$$\mathbf{F}_V = \int_{V_M(t)} \mathbf{f} dV \tag{2.10}$$

In addition to the body forces acted directly on the fluid mass, forces that the viscous stress and pressure exerted on the surface of the control volume must also be considered. If **n** is the normal unit vector on the surface $S_M$ pointing towards the surrounding fluid and **s**(**n**) is the force per unit area exerted by the surrounding fluid on the control volume surface, the net surface force on $S_M(t)$ can be written as follows.

$$\mathbf{F}_s = \int_{S_M(t)} \mathbf{s}(\mathbf{n}) dS \tag{2.11}$$

It is well established that, due to stress equilibrium on any vanishingly small volume, the total stress, **s**(**n**), can be calculated as a dot product between **n** and the *stress tensor* $(\boldsymbol{\sigma})$ as shown below (Deen 1998).

$$\mathbf{s}(\mathbf{n}) = \mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{n} \cdot \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \tag{2.12}$$

The element of the stress tensor is simply the force per unit area. In this book, the notation is chosen such that $\sigma_{xy}$ is the force per unit area in the *y*-direction acting on a surface with the normal vector pointing in the *x*-direction; the first alphabet in the subscript indicates the direction of the normal vector on the surface whereas the second alphabet indicates the direction of the force[3]. The contributions to **σ** are often divided into the contribution from the pressure and that from the *viscous stress tensor* as follows.

$$\boldsymbol{\sigma} = -P\boldsymbol{\delta} + \boldsymbol{\Pi} \tag{2.13}$$

where *P* is the *pressure* and **δ** is the identity matrix; the pressure force is always perpendicular to $S_M$. **Π** is the *viscous stress tensor* that is sometimes referred to as the *deviatoric stress tensor*; its element is the viscous force per unit area. In keeping with the notation employed for **σ**, $\Pi_{xy}$, is the viscous force in the *y*-direction (per unit area) acting on a surface with the normal

[3] It is worth noting that, in other textbooks, a slightly different notation is sometimes employed where the total stress, **s(n)**, is defined as **σ·n** with **σ** being a stress tensor. Its element is defined differently. For instance, $\sigma_{xy}$, is a force per unit surface area in the *x*-direction exerted on a surface with the normal unit vector pointing in the *y*-direction. For most fluids, **σ** is symmetric, causing both this notation and the notation employed in the present book to result in a similar **σ**.

vector pointing in the $x$-direction. (The first alphabet in the subscript indicates the direction of the normal vector on the surface whereas the second alphabet indicates the direction of the viscous force.) Due to conservation of angular momentum, for most fluids, $\mathbf{\Pi}$ is found to be symmetric. There are exceptions where the viscous stress tensor is found to be asymmetric such as those of flows of fluids containing suspensions of colloidal magnetites. For majority of fluids in the liquid and gas phases, however, $\mathbf{\Pi}$ is generally a symmetric tensor, and subsequently, so is $\boldsymbol{\sigma}$.

Substituting the expressions for body forces and stresses stated in Eqs. (2.9)-(2.13) into Eq. (2.7), one obtains the relationship between the change in linear momentum and the forces exerted on $V_M$ as shown below.

$$\int_{V_M(t)} \rho \frac{D\mathbf{v}}{Dt} dV = \int_{V_M(t)} \rho \mathbf{g} dV + \int_{S_M(t)} \mathbf{n} \cdot \left(-P\boldsymbol{\delta} + \mathbf{\Pi}\right) dS + \int_{V_M(t)} \mathbf{f} dV \tag{2.14}$$

If the divergence theorem is applied to change the surface integral in Eq. (2.14) into a volume integral, in the limit of an infinitesimal $V_M$, Eq. (2.14) can be expressed as follows.

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla P + \nabla \cdot \mathbf{\Pi} + \mathbf{f} \tag{2.15}$$

Equation (2.15) is referred to as the *Cauchy momentum equation* and is the governing equation for pressure and velocity of fluids in the systems with length scales much larger than that of fluid molecules such that continuum mechanics is applicable. What remains to be discussed is the value of the viscous stress tensor. As discussed further below in Sec. 2.4, the viscous stress tensor and the "rate of deformation" or the strain rate within the fluid flow are related through the constitutive equation that depends on the molecular interaction between fluid molecules.

## 2.4 Viscous stress tensor as a function of the rate–of-strain tensor in Newtonian fluid flows.

This book focuses primarily on behaviors of confined flows of incompressible *Newtonian fluids*, fluids that respond very quickly to the change in the strain rate in the flows. The magnitude of the element of the Newtonian fluid viscous stress tensor is linearly proportional to the rate of strain tensor and can be expressed as shown below.

$$\mathbf{\Pi} = 2\mu \mathbf{\Gamma} + \left(\aleph - 2\mu/3\right)\left(\nabla \cdot \mathbf{v}\right)\boldsymbol{\delta} \tag{2.16}$$

where $\mu$ is the fluid shear viscosity, a macroscopic manifestation of interactions between fluid molecules. $\aleph$ is known as the dilational viscosity. $\mathbf{\Gamma}$ is the *rate-of-strain tensor* corresponding to the rate of deformation caused by the non-uniformity of fluid velocity expressed as follows.

$$\mathbf{\Gamma} = \frac{1}{2}\left(\nabla \mathbf{v} + \nabla \mathbf{v}^T\right) \tag{2.17}$$

where $\nabla\mathbf{v}$ denotes the *velocity gradient* written as shown below.

$$\nabla\mathbf{v}=\begin{bmatrix}\frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x}\\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y}\\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z}\end{bmatrix} \tag{2.18}$$

The velocity gradient (as a tensor) is discussed in detail in Sec. A2 of the appendix. Whereas the gradient of a scalar function is a vector, the gradient of a vector is a tensor; as shown in Eq. (2.18), the first, second and third columns of the velocity gradient are simply the gradient of the velocity in the *x-, y-* and *z-* directions, respectively. $\nabla\mathbf{v}^T$ appeared earlier in Eq. (2.17) is the transpose of $\nabla\mathbf{v}$. The difference between the fluid velocity at two different locations can be approximated as the product between the velocity gradient and the displacement between the two locations;

$$\Delta\mathbf{v}=\mathbf{v}(\mathbf{r}_2)-\mathbf{v}(\mathbf{r}_1)\approx(\Delta\mathbf{r}\cdot\nabla\mathbf{v}) \tag{2.19}$$

where $\mathbf{v}(\mathbf{r}_1)$ and $\mathbf{v}(\mathbf{r}_2)$ are the fluid velocity at the locations $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. The rate of deformation can be examined from the rate of change of $|\Delta\mathbf{r}|^2$ as follows.

$$\frac{d|\Delta\mathbf{r}|^2}{dt}=2\Delta\mathbf{v}\cdot\Delta\mathbf{r}\approx 2\Delta\mathbf{r}\cdot\nabla\mathbf{v}\cdot\Delta\mathbf{r}=2\Delta\mathbf{r}\cdot\boldsymbol{\Gamma}\cdot\Delta\mathbf{r} \tag{2.20}$$

where $\boldsymbol{\Gamma}$ is the rate-of-strain tensor defined in Eq. (2.17). $\Delta\mathbf{r}\cdot\nabla\mathbf{v}\cdot\Delta\mathbf{r}=\Delta\mathbf{r}\cdot\boldsymbol{\Gamma}\cdot\Delta\mathbf{r}$ because $\Delta\mathbf{r}\cdot\left(\nabla\mathbf{v}-\nabla\mathbf{v}^T\right)\cdot\Delta\mathbf{r}=0$. (The detailed proof is given in Sec. A2 of the appendix.) Worthy of note is the fact that $\boldsymbol{\Gamma}$ is a symmetric tensor. Equation (2.20) demonstrates that the rate of deformation (or the rate of change of the displacement between two material points) in a fluid flow depends only on the symmetric part of the velocity gradient[4]. The relationship between $\boldsymbol{\Pi}$ and $\boldsymbol{\Gamma}$ in Eq. (2.16) is referred to as the *Newtonian hypothesis*: a first-order correction to the theory of *perfect fluids* (or *inviscid fluids*) in the limit where all gradients are small. It is applicable to real fluids in the limit of small fluid velocity (Happel and Brenner, 1983). For non-Newtonian fluids where the slow response to the change in the rate of strain results in either the shear viscosity dependence on the strain rate or the "memory" effect on the relationship between the stress and the rate of strain becoming not negligible, the behavior of fluid velocity and pressure is governed by the Cauchy momentum equation where the relationship between $\boldsymbol{\Pi}$, $\mu$ and $\boldsymbol{\Gamma}$ must be determined from the constitutive equation. Examples of such fluids include flows of polymer melts and solutions containing strongly interacting particles.

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[4] The velocity gradient can also be defined in a slightly different way where its element, $\nabla v_{ij}$, is $\partial v_i/\partial x_j$. This results in slightly different expressions for Eqs. (2.19) and (2.20) as discussed in the appendix. However, both ways that the velocity gradient are defined result in $\frac{d|\Delta\mathbf{r}|^2}{dt}\approx 2\Delta\mathbf{r}\cdot\boldsymbol{\Gamma}\cdot\Delta\mathbf{r}$; the rate of the deformation is dependent on the symmetric $\boldsymbol{\Gamma}$ only.

## 2.5 Navier-Stokes equation: a governing equation for the pressure and velocity of incompressible Newtonian fluid flows

Fluids with constant densities are often called *incompressible fluids* due to the fact that it is "nearly" impossible to reduce the fluid volume if its mass is kept constant. Because the density of any pure fluid is dependent on pressure and temperature, *incompressibility* is an "approximation"; the term "incompressible fluids" is used in referring to fluids with mass per unit volume remaining unaltered in an "approximate" sense. For instance, the sixty times change in an air density (at room temperature) can be caused by a change in pressure from 1 atm to 50 atm, whereas the change in water density, under the same circumstance, is less than 1%. Water is, therefore, classified as an incompressible fluid. Gases are, on the other hand, generally compressible. However, if the gas velocity is much smaller than the sound speed, its density at constant temperature can be approximated as being constant, and an assumption of incompressibility becomes a good approximation.

If, under the circumstance of the system of interest, the fluid density, $\rho$, can be approximated as unchanging, for an infinitesimal $V_M$, Eq. (2.6) can be rewritten as

$$\nabla \cdot \mathbf{v} = 0 \tag{2.21}$$

Eq. (2.21) is known as the *continuity equation*. The fluid velocity with its divergence equaling zero is a particular characteristic of incompressible fluid flows; the physical interpretation is that the total fluid volume and mass fluxes across any closed surfaces must be zero. Newtonian fluids with constant densities are referred to as *incompressible Newtonian fluids*. According to Eq. (2.16), as the divergence of the fluid velocity vanishes in the case of an incompressible Newtonian fluid, $\mathbf{\Pi} = 2\mu\mathbf{\Gamma}$, and its divergence is

$$\nabla \cdot \mathbf{\Pi} = \nabla \cdot \left(2\mu\mathbf{\Gamma}\right) = \mu\nabla^2\mathbf{v} \tag{2.22}$$

As a result, the equation of motion stated in Eq. (2.15) can be expressed as

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla\wp + \mu\nabla^2\mathbf{v} + \mathbf{f} \tag{2.23}$$

where $\wp$ is the *dynamic pressure* defined as

$$\nabla\wp = \nabla P - \rho\mathbf{g} \tag{2.24}$$

The spatial variation of the dynamic pressure is associated with fluid motion; if $\mathbf{f}$ is absent and the fluid is stationary, $\nabla\wp = 0$.

The governing equation for dynamic pressure and velocity of incompressible Newtonian fluids, Eq. (2.23), is referred to as the *Navier-Stokes equation*, due to the fact that it was proposed separately by L. Navier in 1822 and G.G. Stokes in 1845. The term on the left-hand side of the equation is simply the material derivative of the fluid linear momentum per unit volume, whereas the first, second and third terms on the right-hand side correspond to the contributions to $\rho\left(D\mathbf{v}/Dt\right)$ due to dynamic pressure gradient, viscous dissipation, and the body force due to additional fields, respectively. Applicable to incompressible and isothermal

Newtonian fluids with constant density and shear viscosity, the Navier-Stokes equation is, in fact, three partial differential equations with four unknowns: the $x$-, $y$- and $z$- components of the fluid velocity ($\mathbf{v}$) as well as the dynamic pressure ($\wp$). It must, therefore, be solved simultaneously with the continuity equation, Eq. (2.21), being the fourth equation. In order to determine the velocity and dynamic pressure of an incompressible Newtonian fluid, the solution of the Navier-Stokes equation (Eq. 2.23) and the continuity equation (Eq. 2.21) that satisfies the appropriate boundary conditions must be obtained. The most commonly known boundary condition is the *no-slip condition* originated from an empirical observation that is verified countless numbers of times. It concerns the matching of the tangential velocity at the interface between two materials as follows.

$$\mathbf{t}\cdot\mathbf{v}\big|_1 = \mathbf{t}\cdot\mathbf{v}\big|_2 \tag{2.25}$$

where $\mathbf{t}$ is the unit vector tangent to the interface. For instance, at an interface between a fluid and an immobile solid, the tangential component of $\mathbf{v}$ at the interface equals zero if the solid surface is inert and smooth.

In addition to the no-slip condition, sometimes other appropriate boundary conditions (depending on the nature of the interface as well as those of the two materials) are also required. For example, $\mathbf{v}$ at the interface between a fluid and an impermeable solid must satisfy the *no-penetration condition*; $\mathbf{n}\cdot\mathbf{v} = 0$. The boundary condition between two fluids is relatively more complicated as it involves the interfacial stress balance. This book, however, focuses on the motion of confined flows and hindered solid particles with the boundary conditions at the solid-fluid interfaces being the no-slip condition and the no-penetration condition.

## 2.6 Static fluids and buoyancy force

If the fluid is completely static ($\mathbf{v} = 0$), both the Cauchy momentum equation and the Navier-Stokes equation become

$$\nabla\wp = \nabla P - \rho\mathbf{g} = 0 \tag{2.26}$$

This expression indicates that the hydrostatic pressure changes only in the direction parallel to the gravitational acceleration ($\mathbf{g}$). For an incompressible static fluid, one obtains a following expression for pressure by integrating once.

$$P(\mathbf{r}) = P(\mathbf{r_0}) + \rho\mathbf{g}\cdot\mathbf{r} \tag{2.27}$$

Another consequence of the fact that the hydrostatic pressure increases with depth is the net pressure force on a completely submerged solid object in a static fluid as shown below.

$$\mathbf{F_P} = -\int_S P\mathbf{n}dS = -\int_V \nabla P dV = -\int_V \rho\mathbf{g}dV \tag{2.28}$$

where $V$ and $S$ is the volume and surface of the solid object, respectively. As shown above, by applying the divergence theorem, it can be demonstrated that the upwards force on a solid object due to the hydrostatic pressure is the weight of a fluid of the same volume, regardless of the solid object shape. This is commonly known as *Archimedes' Principle*. The net force on the submerged object, the difference between its weight and $\mathbf{F_P}$, is referred to as the buoyancy force ($\mathbf{F_B}$); for a solid object with constant density ($\rho_s$), it can be expressed as shown below.

$$\mathbf{F_B} = \rho_s \mathbf{g} V - \mathbf{F_P} = (\rho_s - \rho)\mathbf{g} V \tag{2.29}$$

It is worth noting that the expressions for the pressure given in Eq. (2.27) and the buoyancy force in Eq. (2.29) is applicable for the solid object surrounded by the static fluid only. If $\mathbf{v} \neq 0$, the total stress exerted on the solid object submerged in a fluid flow must be computed as an addition of the pressure and viscous stress obtained from the solution of either the Cauchy momentum equation (for any fluid with a known constitutive equation) or the Navier-Stokes equation (if the fluid is an incompressible Newtonian fluid).

## 2.7 Generalized Newtonian fluids

As aforementioned, the Navier-Stokes equation is a governing equation for the velocity and pressure of incompressible Newtonian fluids. Any fluid that does not obey Eq. (2.16) is known as a *non-Newtonian fluid*. To calculate its velocity and pressure, the Cauchy momentum equation must be solved in conjunction with the continuity equation. To illustrate the difference between Newtonian and non-Newtonian fluids, a special category of non-Newtonian fluids, the *generalized Newtonian fluids*, is considered. A generalized Newtonian fluid differs from a Newtonian fluid because the viscous stress tensor ($\mathbf{\Pi}$) of the generalized Newtonian fluid is not a linear function of the rate-of-strain tensor ($\mathbf{\Gamma}$). In other words, the fluid viscosity is not independent of $\mathbf{\Gamma}$ (Deen, 1998), and an empirical expression is often employed in describing their relationship. For example, the viscosity of many polymeric liquids is observed to be constant at a low shear rate but becomes dependent on the strain rate as the shear rate increases before reaching a plateau and becoming constant once again. These polymeric liquids behave like Newtonian fluids in the limit of very low and very high shear rates. At an intermediate shear rate, however, the empirical expression used in describing the relationship between $\mu$ and $\mathbf{\Gamma}$ is a power-law model as follows.

$$\mu = \Upsilon (2\Gamma)^{n-1} \tag{2.30}$$

where $\Gamma = (1/2)(\mathbf{\Gamma} : \mathbf{\Gamma})^{1/2} = (1/2)\left(\sum_i \sum_j \Gamma_{ij}\Gamma_{ji}\right)^{1/2}$. $\Upsilon$ and $n$ are constants. Fluids that obey Eq. (2.30) is often called a *power-law fluid*. If $n < 1$, the fluid is referred to as the *shear-thinning fluid* or the *pseudo-plastic fluid*. Examples of shear thinning fluids include a hair-styling gel that produces little resistance as it is rubbed between fingers. Any fluid that obeys Eq. (2.30) but with $n > 1$, on the other hand, is known as the *shear-thickening fluid* or the *dilatant fluid*. If $n = 1$, the fluid behavior is that of a Newtonian fluid and $\Upsilon$ is simply $\mu$. Equation (2.30) is not the only empirical expression employed in relating $\mu$ and $\mathbf{\Gamma}$ of generalized Newtonian fluids. For instance, a *Carreau model* is an expression that is equivalent to Eq. (2.30) but contains four instead of two constants and encompasses the Newtonian behavior at very low and very high shear rates and is also employed in a constitutive equation for polymeric liquids.

## 2.8 Analysis of confined fully developed unidirectional flows

To familiarize the readers with the Cauchy momentum equation, the Navier-Stokes equation and the continuity equation as governing equations for the velocity and pressure of incompressible fluids, a calculation of a fluid velocity of a steady fully developed unidirectional flow in a microchannel is first considered. Because there is only one non-zero velocity component, the inertial term of the Navier-Stokes equation $(\rho \mathbf{v} \cdot \nabla \mathbf{v})$ always vanishes (as proven in Sec. A5 in the appendix) resulting in the equation becomes automatically linearized. In addition, in the following examples, other volumetric body forces (apart from the gravitational force) are absent; $\mathbf{f} = 0$. Examples presented in this section include an analytically obtained relationship between the dynamic pressure gradient and the velocity of a fully developed flow between parallel infinite plates as well as a relationship between the average fluid speed and the imposed pressure difference for a flow confined in a long channel with a rectangular cross-section, followed by a discussion of the limit of the assumption of the flow being fully developed.

*2.8.1 A pressure driven power-law fluid flow between parallel plates*

To illustrate the difference between a Newtonian fluid and a generalized Newtonian fluid, a fully developed unidirectional steady flow of a power-law fluid between parallel square plates is considered. As shown in Fig. 2.1, $H$ is the half-width of the gap between the plates. The flow is generated by the applied pressure gradient in the $z$-direction. The size of the plate is much larger than $H$ such that, beyond a certain distance from the edges of the plates, the flow is fully developed and unidirectional; $\mathbf{v} = v_z(x)\mathbf{e}_z$ with $v_y = v_z = 0$. As a result, according to Eq. (2.23), $\partial \wp / \partial y = \partial \wp / \partial z = 0$, and, $\wp = \wp(x)$.

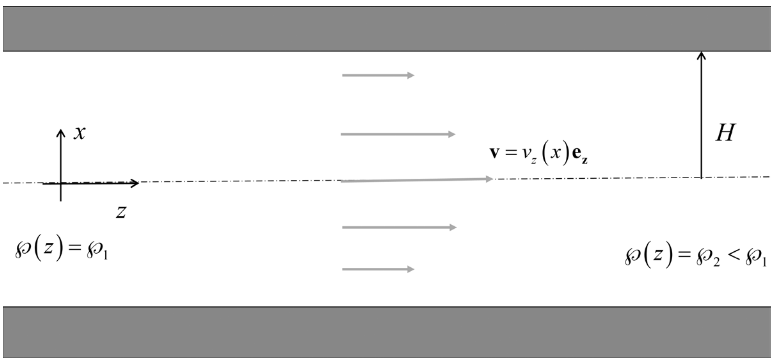


**Figure 2.1:** Schematic drawing of a fully developed unidirectional flow of a power-law fluid confined between parallel infinite plates. $H$ is the half-width of the gap between the plates. The $z$-direction is the direction of the generated flow, whereas the $x$-direction is the transverse direction. $\mathbf{v}$ is the fluid velocity, whereas $\wp(z)$ is the dynamic pressure with $\wp_1$ and $\wp_2$ being the upstream and downstream dynamic pressure, respectively.

As the fluid between the plates is a generalized Newtonian fluid and not a Newtonian fluid, the Cauchy momentum equation must be employed. The flow is assumed to be steady and fully developed, allowing the Cauchy momentum equation to be re-written as

$$0=-\frac{\partial \wp}{\partial z}+\frac{\partial \Pi_{xz}(x)}{\partial x} \tag{2.31}$$

where $\Pi_{xz}$ is the only non-zero element of the viscous stress tensor. According to the power law model stated in Eq. (2.30), it can be expressed as

$$\Pi_{xz}=-\Upsilon\left|\frac{dv_z}{dx}\right|^n \tag{2.32}$$

Substituting the expression for $\Pi_{xz}$ stated in Eq. (2.32) into Eq. (2.31), one obtains

$$\frac{dv_z}{dx}=-\left(-\frac{1}{\Upsilon}\frac{d\wp}{dz}\right)^{1/n}x^{1/n} \tag{2.33}$$

Integrating Eq. (2.33) once and using the no-slip boundary condition at the solid surfaces $\left(v_z\left(x=\pm H\right)=0\right)$, the fluid velocity is found to be

$$v_z(x)=\left(-\frac{1}{\Upsilon}\frac{d\wp}{dz}\right)^{1/n}\left(\frac{n}{n+1}\right)(H)^{\frac{n+1}{n}}\left(1-\left(\frac{|x|}{H}\right)^{\frac{n+1}{n}}\right) \tag{2.34}$$

The fluid velocity profile is shown in Fig. 2.2 for the case of a shear-thinning fluid ($n$ = 0.5), a Newtonian fluid ($n$ = 1) and a shear-thickening fluid ($n$ = 2), respectively. If the fluid between the parallel plates is a Newtonian fluid, $n$ = 1 and $\Upsilon=\mu$, resulting in $v_z(x)$ being simply

$$v_z(x)=-\frac{H^2}{2\mu}\frac{d\wp}{dz}\left(1-\left(\frac{x}{H}\right)^2\right) \tag{2.35}$$

The Newtonian fluid flow with a parabolic velocity profile expressed in Eq. (2.35) and shown in Fig. 2.2 (as a line with square symbols) is referred to as a *plane Poiseuille flow*.

Often, the experimentally measured quantity is not $v_z(x)$ but its average $\left(\langle v_z\rangle\right)$ obtained from the cross-sectional average of $v_z(x)$ as follows.

$$\langle v_z\rangle=\frac{-H^2}{3\mu}\left(\frac{d\wp}{dz}\right) \tag{2.36}$$

The fluid velocity of the plane Poisseuile flow can then be rewritten as

$$v_z(x)=\frac{3}{2}\langle v_z\rangle\left(1-\left(\frac{x}{H}\right)^2\right) \tag{2.37}$$